Teze disertace
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Steady compressible Navier–Stokes–Fourier system and related problems. Large data results.

Komise pro obhajoby doktorských prací v oboru Matematická analýza a příbuzné obory

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1 Introduction: Compressible heat conducting Newtonian fluid

We shall briefly introduce the models coming from the continuum mechanics and thermodynamics which we study later. More detailed information can be found, e.g., in the monographs [Gurtin 1991], [Gallavotti 2002] or [Lamb 1993].

We consider the three fundamental balance laws: the balance of mass, the balance of linear momentum and the balance of total energy. Using the so-called Eulerian description (which is commonly used for equations of fluid dynamics) we have in $(0,T)\times\Omega$

$$\begin{aligned} \frac{\partial\varrho}{\partial t} + \operatorname{div}(\varrho\mathbf{u}) &= 0, \\ \frac{\partial(\varrho\mathbf{u})}{\partial t} + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbb{T} &= \varrho\mathbf{f}, \\ \frac{\partial(\varrho E)}{\partial t} + \operatorname{div}(\varrho E\mathbf{u}) + \operatorname{div}\mathbf{q} - \operatorname{div}(\mathbb{T}\mathbf{u}) &= \varrho\mathbf{f}\cdot\mathbf{u}. \end{aligned} \tag{1.1}$$

The classical formulation of these equations is actually not what we are going to deal with in this thesis. We shall work with weak or variational entropy solutions. These formulations, stated later in the thesis, can be derived directly from the integral formulation of the balance laws. Therefore we do not need to work with the classical formulation of the balance laws, however, it is common to formulate the balance laws in the differential form even though their weak formulation is considered.

For simplicity, we assume that the spatial domain $\Omega\subset\mathbb{R}^N$, $N=2$ or 3, is bounded and fixed. We shall mostly deal with the case $N=3$, however, in some cases we also consider $N=2$. Above, ϱ: $(0,T)\times\Omega\to\mathbb{R}^+$ is the density of the fluid, $\mathbf{u}$: $(0,T)\times\Omega\to\mathbb{R}^N$ is the velocity, E: $(0,T)\times\Omega\to\mathbb{R}^+$ is the specific total energy , $\mathbb{T}$: $(0,T)\times\Omega\to\mathbb{R}^{N\times N}$ is the stress tensor, $\mathbf{q}$: $(0,T)\times\Omega\to\mathbb{R}^N$ is the heat flux, and the given vector field $\mathbf{f}$: $(0,T)\times\Omega\to\mathbb{R}^N$ denotes the external volume force. Recall that $E=\frac{1}{2}|\mathbf{u}|^2+e$, where $\frac{1}{2}|\mathbf{u}|^2$ is the specific kinetic energy and e is the specific internal energy. Generally, the balance of the angular momentum should also be taken into account together with (1.1). However, if we do not assume any internal momenta of the continuum, it can be verified that as a consequence of the angular momentum balance the stress tensor $\mathbb{T}$ must be symmetric which we assume in what follows.

We take (as commonly used) for our basic thermodynamic quantities the density ϱ and the thermodynamic temperature ϑ. Therefore all other quantities, i.e., the stress tensor $\mathbb{T}$, the specific internal energy e and the heat flux $\mathbf{q}$ are given functions of t, x, ϱ, $\mathbf{u}$ and ϑ. The standard assumptions from the continuum mechanics yield that

$$\mathbb{T} = -p(\varrho,\vartheta)\mathbb{I} + \mathbb{S}(\varrho,\mathbb{D}(\mathbf{u}),\vartheta), \tag{1.2}$$

where $\mathbb{I}$ denotes the unit tensor, the scalar quantity p (a given function of the density and temperature) is the pressure, $\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the symmetric part of the velocity gradient and the tensor $\mathbb{S}$ is the viscous part of the stress tensor. We consider only linear dependence of the stress tensor on the symmetric part of the velocity gradient. This, together with the assumption that the viscosities are density independent leads to

$$\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta) = \mu(\vartheta)\Big(2\mathbb{D}(\mathbf{u}) - \frac{2}{N}\operatorname{div}\mathbf{u}\,\mathbb{I}\Big) + \xi(\vartheta)\operatorname{div}\mathbf{u}\,\mathbb{I}. \tag{1.3}$$

The scalar functions $\mu(\cdot) > 0$ and $\xi(\cdot) \geq 0$ are called the shear and the bulk viscosities. We shall study the situations with $\mu(\vartheta) \sim (1+\vartheta)^a$ a Lipschitz continuous function and $\xi(\vartheta) \leq C(1+\vartheta)^a$ a continuous function for $0 \leq a \leq 1$ and $C > 0$. For the pressure, we mostly consider the gas law of the form

$$p(\varrho,\vartheta) = (\gamma-1)\varrho e(\varrho,\vartheta), \tag{1.4}$$

a generalization of the law for the monoatomic gas, where $\gamma = \frac{5}{3}$. In general, the value $\frac{5}{3}$ is the highest physically interesting value and for all other gases we should take $1 \leq \gamma \leq \frac{5}{3}$, cf. [Elizier et al 1996].

We also sometimes replace assumption (1.4) by

$$p(\varrho,\vartheta) = \varrho^\gamma + \varrho\vartheta, \quad e(\varrho,\vartheta) = \frac{1}{\gamma-1}\varrho^{\gamma-1} + c_v\vartheta, \text{ with } c_v > 0, \tag{1.5}$$

whose physical relevance is discussed in [Feireisl 2004]. The pressure and the specific internal energy from (1.5) are in fact a simplification of (1.4) which still contains the same asymptotic properties and hence also leads to the same main mathematical difficulties as the more general model (1.4).

The heat flux is assumed to fulfil the Fourier law

$$\mathbf{q} = \mathbf{q}(\vartheta,\nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta \tag{1.6}$$

with the heat conductivity $\kappa(\vartheta) \sim (1+\vartheta)^m$ for some $m > 0$.

To get a well posed problem, we must prescribe the initial conditions

$$\varrho(0,x) = \varrho_0(x), \quad (\varrho\mathbf{u})(0,x) = \mathbf{m}_0(x), \quad \vartheta(0,x) = \vartheta_0(x) \tag{1.7}$$

in Ω and the boundary conditions on $\partial\Omega$. We restrict ourselves to the following simple cases. For the heat flux, we take

$$-\mathbf{q}\cdot\mathbf{n} + L(\vartheta)(\vartheta - \Theta_0) = 0 \tag{1.8}$$

and for the velocity we consider either the homogeneous Dirichlet boundary conditions

$$\mathbf{u} = \mathbf{0} \tag{1.9}$$

or the (partial) slip boundary conditions (sometimes also called the Navier boundary conditions)

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad (\mathbb{S}\mathbf{n})\times\mathbf{n} + \alpha\mathbf{u}\times\mathbf{n} = \mathbf{0}. \tag{1.10}$$

Above, $\mathbf{n}$ denotes the external normal vector to $\partial\Omega$, Θ_0: $(0,T)\times\partial\Omega \to \mathbb{R}^+$ is the external temperature, $L(\vartheta) \sim (1+\vartheta)^l$, a continuous function, characterizes the thermal insulation of the boundary, and $\alpha \geq 0$ is the friction coefficient which is for simplicity assumed to be constant. Since in what follows we consider only the steady or time-periodic problems, we cannot assume the boundary to be at the same time thermally (i.e. zero heat flux) and mechanically insulated as the set of such solutions would be quite trivial, cf. [Feireisl Pražák 2010].

The Second law of thermodynamics implies the existence of a differentiable function $s(\varrho,\vartheta)$ called the specific entropy which is (up to an additive constant) given by the Gibbs relation

$$\frac{1}{\vartheta}\Big(\mathbf{D}e(\varrho,\vartheta) + p(\varrho,\vartheta)\mathbf{D}\Big(\frac{1}{\varrho}\Big)\Big) = \mathbf{D}s(\varrho,\vartheta).$$

Due to (1.4) and (1.1), it is not difficult to verify, at least formally, that the specific entropy obeys the entropy equation

$$\frac{\partial(\varrho s)}{\partial t} + \operatorname{div}(\varrho s\mathbf{u}) + \operatorname{div}\Big(\frac{\mathbf{q}}{\vartheta}\Big) = \frac{\mathbb{S}:\nabla\mathbf{u}}{\vartheta} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2}. \tag{1.11}$$

On this level, equation (1.11) is fully equivalent with the total energy equality $(1.1)_3$ and can replace it. Another equivalent formulation is the internal energy balance in the form

$$\frac{\partial(\varrho e)}{\partial t} + \operatorname{div}(\varrho e\mathbf{u}) + \operatorname{div}\mathbf{q} + p\operatorname{div}\mathbf{u} = \mathbb{S} : \nabla\mathbf{u}.$$

It can be deduced easily from the total energy balance $(1.1)_3$ subtracting the kinetic energy balance, i.e., $(1.1)_2$ multiplied by $\mathbf{u}$. Indeed, at the level of classical solutions such computations are possible; later on, on the level of weak solutions, these formulations may not be equivalent.

It is also easy to verify that the functions p and e are compatible with the existence of entropy if and only if they satisfy the Maxwell relation

$$\frac{\partial e(\varrho,\vartheta)}{\partial\varrho} = \frac{1}{\varrho^2}\Big(p(\varrho,\vartheta) - \vartheta\frac{\partial p(\varrho,\vartheta)}{\partial\vartheta}\Big). \tag{1.12}$$

Note that the choice (1.5) fulfils it. Assuming relation (1.4), if the pressure function $p \in C^1((0,\infty)^2)$, then it has necessarily the form

$$p(\varrho,\vartheta) = \vartheta^{\frac{\gamma}{\gamma-1}} P\Big(\frac{\rho}{\vartheta^{\frac{1}{\gamma-1}}}\Big), \tag{1.13}$$

where $P \in C^1((0,\infty))$.

We shall assume that

$$\begin{gathered} P(\cdot) \in C^1([0,\infty)) \cap C^2((0,\infty)), \\ P(0) = 0, \quad P'(0) = p_0 > 0, \quad P'(Z) > 0, \quad Z > 0, \\ \lim_{Z\to\infty}\frac{P(Z)}{Z^\gamma} = p_\infty > 0, \\ 0 < \frac{1}{\gamma-1}\frac{\gamma P(Z) - ZP'(Z)}{Z} \le c_7 < \infty, \quad Z > 0. \end{gathered} \tag{1.14}$$

For more details about (1.4) and about physical motivation for assumptions (1.14) see e.g. [Feireisl Novotný 2009, Sections 1.4.2 and 3.2].

In what follows we shall now deal with the steady version of system (1.1), i.e., we consider the following system of equations

$$\begin{aligned} \operatorname{div}(\varrho\mathbf{u}) &= 0, \\ \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbb{S} + \nabla p &= \varrho\mathbf{f}, \\ \operatorname{div}(\varrho E\mathbf{u}) + \operatorname{div}\mathbf{q} - \operatorname{div}(\mathbb{S}\mathbf{u}) + \operatorname{div} p\mathbf{u} &= \varrho\mathbf{f}\cdot\mathbf{u} \end{aligned} \tag{1.15}$$

with the boundary conditions (1.8) and either (1.9) or (1.10). The initial conditions are not relevant in this case, on the other hand, we must prescribe the total mass of the fluid

$$\int_\Omega \varrho \, \mathrm{d}x = M > 0. \tag{1.16}$$

Recall that, on the level of smooth solutions with strictly positive temperature, the total energy balance (together with the momentum balance and the continuity equation) is equivalent with either the internal energy balance

$$\operatorname{div}(\varrho e \mathbf{u}) + \operatorname{div} \mathbf{q} + p \operatorname{div} \mathbf{u} = \mathbb{S} : \nabla \mathbf{u} \tag{1.17}$$

or the entropy balance

$$\operatorname{div}(\varrho s \mathbf{u}) + \operatorname{div}\Big(\frac{\mathbf{q}}{\vartheta}\Big) = \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2}; \tag{1.18}$$

in both cases, we must consider them together with the momentum balance and the continuity equation.

We aim at studying existence of solutions to this problem without any restriction on the size of the data. It requires that we deal with the weak solutions (or even weaker notion). These solutions may be non-unique and their existence may depend on

- the values of the parameters γ and m
- the choice of the boundary conditions (1.9) or (1.10)
- the type of the solutions (as explained in the next section).

2 Mathematical theory for steady compressible Navier–Stokes–Fourier system

2.1 Definitions of solutions for different formulations

Our aim is to study the existence of solutions without any restriction on the size of the data and keep the regularity assumptions on the data as general as possible. This leads us naturally to the notion of weak solution (or, as explained below, variational entropy solution). Before dealing with the formulations allowing very low exponent γ, we introduce a definition based on the internal energy balance, where we can obtain relatively regular solutions for a certain range of γ. We consider the Navier boundary conditions (1.10) for the velocity, assume the viscosities to be constant (i.e., we take $a = 0$ below (1.3)) and use the pressure law (1.5).

We use standard notation for the functions spaces (Lebesgue, Sobolev or spaces of continuous or continuously differentiable functions). We further denote

$$W^{1,p}_{\mathbf{n}}(\Omega;\mathbb{R}^3) = \{\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^3); \mathbf{u}\cdot\mathbf{n} = 0 \text{ in the sense of traces}\}.$$

Similarly the space $C^1_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^3)$ contains all differentiable functions up to the boundary of Ω with zero normal trace on $\partial\Omega$. Then we have

Definition 1 (Weak solution for internal energy formulation.) *The triple $(\varrho, \mathbf{u}, \vartheta)$ is called a weak solution to system (1.15)$_{1-2}$, (1.17), (1.16), (1.3), (1.5), (1.6), (1.8) and (1.10) if $\varrho \in L^{\frac{6\gamma}{5}}(\Omega)$, $\mathbf{u} \in W^{1,2}_{\mathbf{n}}(\Omega;\mathbb{R}^3)$, $\vartheta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega) \cap L^{l+1}(\partial\Omega)$, $r > 1$ with $\varrho|\mathbf{u}|^2 \in L^{\frac{6}{5}}(\Omega)$, $\varrho\mathbf{u}\vartheta \in L^1(\Omega;\mathbb{R}^3)$, $\mathbb{S}(\mathbb{D}(\mathbf{u})) : \mathbb{D}(\mathbf{u}) \in L^1(\Omega)$, $\vartheta^m\nabla\vartheta \in L^1(\Omega;\mathbb{R}^3)$. Moreover, the continuity equation is satisfied in the weak sense*

$$\int_\Omega \varrho\mathbf{u}\cdot\nabla\psi \,\mathrm{d}x = 0 \qquad \forall\psi \in C^1(\overline{\Omega}), \tag{2.19}$$

the momentum equation holds in the weak sense

$$\begin{aligned}\int_\Omega \big(-\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} - p(\varrho,\vartheta)\operatorname{div}\boldsymbol{\varphi} + \mathbb{S}(\mathbb{D}(\mathbf{u})) : \nabla\boldsymbol{\varphi}\big) \,\mathrm{d}x \\ +\alpha\int_{\partial\Omega} \mathbf{u}\cdot\boldsymbol{\varphi} \,\mathrm{d}S = \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi} \,\mathrm{d}x \quad \forall\boldsymbol{\varphi} \in C^1_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^3),\end{aligned} \tag{2.20}$$

and the internal energy balance holds in the weak sense

$$\begin{aligned}&\int_\Omega \Big(\kappa(\vartheta)\nabla\vartheta - \varrho\vartheta\mathbf{u}\Big)\cdot\nabla\psi \,\mathrm{d}x + \int_{\partial\Omega} L(\vartheta)(\vartheta - \Theta_0)\psi \,\mathrm{d}S \\ &= \int_\Omega \Big(\mathbb{S}(\mathbb{D}(\mathbf{u})) : \nabla\mathbf{u} + \varrho\vartheta\operatorname{div}\mathbf{u}\Big)\psi \,\mathrm{d}x \quad \forall\psi \in C^1(\overline{\Omega}).\end{aligned} \tag{2.21}$$

Note that we used the fact that in the weak formulation of the internal energy balance, the cold pressure terms are cancelled with the cold internal energy terms. This is, at least formally, always true, but it requires certain integrability of the density. Since we deal with this definition only for $\gamma > 3$ later on, these terms cancel even for weak solutions.

Next we consider either the total energy balance formulation (which leads to the weak formulation). The definitions for the Dirichlet and Navier boundary conditions slightly differ, therefore we present both. In both cases, we consider either (1.5) or (1.4) with (1.12)–(1.14) and as above, we must prescribe the total mass (1.16).

Definition 2 (Total energy formulation for Dirichlet b.c.) *The triple $(\varrho, \mathbf{u}, \vartheta)$ is called a weak solution to system (1.15), (1.3), (1.4), (1.6), (1.8), (1.9) and (1.16), if $\varrho \in L^{\frac{6\gamma}{5}}(\Omega)$, $\int_\Omega \varrho \; \mathrm{d}x = M$, $\mathbf{u} \in W_0^{1,2}(\Omega;\mathbb{R}^3)$, $\vartheta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega) \cap L^{l+1}(\partial\Omega)$, $r > 1$ with $\varrho|\mathbf{u}|^2 \in L^{\frac{6}{5}}(\Omega)$, $\varrho\mathbf{u}\vartheta \in L^1(\Omega;\mathbb{R}^3)$, $\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta)\mathbf{u} \in L^1(\Omega;\mathbb{R}^3)$, $\vartheta^m \nabla\vartheta \in L^1(\Omega;\mathbb{R}^3)$, and*

$$\int_\Omega \varrho\mathbf{u}\cdot\nabla\psi \; \mathrm{d}x = 0 \qquad \forall \psi \in C^1(\overline{\Omega}), \tag{2.22}$$

$$\begin{aligned}\int_\Omega \big(-\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} - p(\varrho,\vartheta)\operatorname{div}\boldsymbol{\varphi} + \mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta) : \nabla\boldsymbol{\varphi}\big) \; \mathrm{d}x \\ = \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi} \; \mathrm{d}x \quad \forall \boldsymbol{\varphi} \in C_0^1(\Omega;\mathbb{R}^3),\end{aligned} \tag{2.23}$$

$$\begin{aligned}\int_\Omega -\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\Big)\mathbf{u}\cdot\nabla\psi \; \mathrm{d}x = \int_\Omega \big(\varrho\mathbf{f}\cdot\mathbf{u}\psi + p(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\big) \; \mathrm{d}x \\ - \int_\Omega \big(\big(\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta)\mathbf{u}\big)\cdot\nabla\psi + \kappa(\cdot,\vartheta)\nabla\vartheta\cdot\nabla\psi\big) \; \mathrm{d}x \\ - \int_{\partial\Omega} L(\vartheta)(\vartheta - \Theta_0)\psi \; \mathrm{d}S \qquad \forall\psi \in C^1(\overline{\Omega}).\end{aligned} \tag{2.24}$$

Definition 3 (Total energy formulation for Navier b.c.) *The triple $(\varrho, \mathbf{u}, \vartheta)$ is called a weak solution to system (1.15), (1.3), (1.4), (1.6), (1.8), (1.10) and (1.16), if $\varrho \in L^{\frac{6\gamma}{5}}(\Omega)$, $\int_\Omega \varrho \; \mathrm{d}x = M$, $\mathbf{u} \in W_{\mathbf{n}}^{1,2}(\Omega;\mathbb{R}^3)$, $\vartheta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega) \cap L^{l+1}(\partial\Omega)$, $r > 1$ with $\varrho|\mathbf{u}|^2 \in L^{\frac{6}{5}}(\Omega)$, $\varrho\mathbf{u}\vartheta \in L^1(\Omega;\mathbb{R}^3)$, $\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta)\mathbf{u} \in L^1(\Omega;\mathbb{R}^3)$, $\vartheta^m \nabla\vartheta \in L^1(\Omega;\mathbb{R}^3)$. Moreover, the continuity equa-*

tion is satisfied in the sense as in (2.19), and

$$\begin{aligned}\int_\Omega \big(-\varrho(\mathbf{u}\otimes\mathbf{u}):\nabla\boldsymbol{\varphi} - p(\varrho,\vartheta)\operatorname{div}\boldsymbol{\varphi} + \mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta):\nabla\boldsymbol{\varphi}\big)\ \mathrm{d}x \\ +\alpha\int_{\partial\Omega}\mathbf{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}S = \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\ \mathrm{d}x \quad \forall\boldsymbol{\varphi}\in C^1_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^3),\end{aligned} \tag{2.25}$$

$$\begin{aligned}\int_\Omega -\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\Big)\mathbf{u}\cdot\nabla\psi\ \mathrm{d}x = \int_\Omega \big(\varrho\mathbf{f}\cdot\mathbf{u}\psi + p(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\big)\ \mathrm{d}x \\ -\int_\Omega \big((\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta)\mathbf{u})\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi\big)\ \mathrm{d}x \\ -\int_{\partial\Omega} L(\vartheta)(\vartheta-\Theta_0)\psi\ \mathrm{d}S - \alpha\int_{\partial\Omega}|\mathbf{u}|^2\psi\ \mathrm{d}S \ \ \forall\psi\in C^1(\overline{\Omega}).\end{aligned} \tag{2.26}$$

Another definition concerns the formulation with the entropy equation. The main problem is that due to mathematical reasons it is difficult to expect that it is possible to obtain equality in the entropy formulation. However, it is enough to prove inequality and in order to keep the weak–strong compatibility (sufficiently smooth solution of this formulation is in fact classical solution to the original formulation), it is necessary to extract at least a part of the information from the total energy balance. Again, formulations for both boundary conditions may include either (1.5) or (1.4) with (1.12)–(1.14).

Definition 4 (Variational entropy solution for Dirichlet b.c.) *The triple $(\varrho,\mathbf{u},\vartheta)$ is called a variational entropy solution to system (1.15)$_{1-2}$, (1.18), (1.3), (1.4), (1.6), (1.8), (1.9) and (1.16), if $\varrho\in L^\gamma(\Omega)$, $\int_\Omega\varrho\ \mathrm{d}x = M$, $\mathbf{u}\in W^{1,2}_0(\Omega;\mathbb{R}^3)$, $\vartheta\in W^{1,r}(\Omega)\cap L^{3m}(\Omega)\cap L^{l+1}(\partial\Omega)$, $r>1$, with $\varrho\mathbf{u}\in L^{\frac{6}{5}}(\Omega;\mathbb{R}^3)$, $\varrho\vartheta\in L^1(\Omega)$, and $\vartheta^{-1}\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta)\mathbf{u}\in L^1(\Omega;\mathbb{R}^3)$, $L(\vartheta)$, $\frac{L(\vartheta)}{\vartheta}\in L^1(\partial\Omega)$, $\kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\in L^1(\Omega)$ and $\kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}\in L^1(\Omega;\mathbb{R}^3)$. Moreover, equalities (2.22) and (2.23) are satisfied in the same sense as in Definition 2, and we have the entropy inequality*

$$\begin{aligned}\int_\Omega\Big(\frac{\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta):\nabla\mathbf{u}}{\vartheta} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\psi\ \mathrm{d}x + \int_{\partial\Omega}\frac{L(\vartheta)}{\vartheta}\Theta_0\psi\ \mathrm{d}S \\ \le \int_{\partial\Omega}L(\vartheta)\psi\ \mathrm{d}S + \int_\Omega\Big(\kappa(\vartheta)\frac{\nabla\vartheta\cdot\nabla\psi}{\vartheta} - \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\Big)\ \mathrm{d}x\end{aligned} \tag{2.27}$$

for all non-negative $\psi \in C^1(\overline{\Omega})$, together with the global total energy balance

$$\int_{\partial\Omega} L(\vartheta)(\vartheta - \Theta_0)\ \mathrm{d}S = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u}\ \mathrm{d}x. \tag{2.28}$$

Similarly as above we have

Definition 5 (Variational entropy solution for Navier b.c.) *The triple $(\varrho, \mathbf{u}, \vartheta)$ is called a variational entropy solution to system (1.15)$_{1-2}$, (1.18), (1.3), (1.4), (1.6), (1.8), (1.10) and (1.16), if $\varrho \in L^\gamma(\Omega)$, $\int_\Omega \varrho\ \mathrm{d}x = M$, $\mathbf{u} \in W^{1,2}_{\mathbf{n}}(\Omega;\mathbb{R}^3)$, $\vartheta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega) \cap L^{l+1}(\partial\Omega)$, $r > 1$, with $\varrho\mathbf{u} \in L^{\frac{6}{5}}(\Omega;\mathbb{R}^3)$, $\varrho\vartheta \in L^1(\Omega)$, $\vartheta^{-1}\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta)\mathbf{u} \in L^1(\Omega;\mathbb{R}^3)$, $L(\vartheta)$, $\frac{L(\vartheta)}{\vartheta} \in L^1(\partial\Omega)$, $\kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2} \in L^1(\Omega)$ and $\kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta} \in L^1(\Omega;\mathbb{R}^3)$. Moreover, equalities (2.22) and (2.25) are satisfied in the same sense as in Definition 3, we have the entropy inequality (2.27) in the same sense as in Definition 4, together with the global total energy balance*

$$\alpha \int_{\partial\Omega} |\mathbf{u}|^2\ \mathrm{d}S + \int_{\partial\Omega} L(\vartheta)(\vartheta - \Theta_0)\ \mathrm{d}S = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u}\ \mathrm{d}x. \tag{2.29}$$

We will also need the notion of the renormalized solution to the continuity equation

Definition 6 (Renormalized solution to continuity equation.) *Let $\mathbf{u} \in W^{1,2}_{loc}(\mathbb{R}^3;\mathbb{R}^3)$ and $\varrho \in L^{\frac{6}{5}}_{loc}(\mathbb{R}^3)$ solve*

$$\operatorname{div}(\varrho\mathbf{u}) = 0 \ \text{in}\ \mathcal{D}'(\mathbb{R}^3).$$

Then the pair $(\varrho, \mathbf{u})$ is called a renormalized solution to the continuity equation, if

$$\operatorname{div}(b(\varrho)\mathbf{u}) + \big(\varrho b'(\varrho) - b(\varrho)\big) \operatorname{div}\mathbf{u} = 0 \ \text{in}\ \mathcal{D}'(\mathbb{R}^3) \tag{2.30}$$

for all $b \in C^1([0,\infty)) \cap W^{1,\infty}((0,\infty))$ with $zb'(z) \in L^\infty((0,\infty))$.

2.2 Main results

2.2.1 Existence of a solution for internal energy formulation

We first describe the result from [Mucha Pokorný 2009] which was the first paper dealing with weak solutions for the steady compressible Navier–Stokes–Fourier system in our formulation. The technique was based on the previous results of both authors, see papers [Mucha Pokorný 2006] and [Pokorný Mucha 2008]. It holds

Theorem 1 (Internal energy formulation.) [**Mucha Pokorný 2009**] *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^3$ which is not axially symmetric if $\alpha = 0$. Let the viscosities be constant. Let $\mathbf{f} \in L^\infty(\Omega; \mathbb{R}^3)$ and*

$$\gamma > 3, \qquad m = l + 1 > \frac{3\gamma - 1}{3\gamma - 7}.$$

Then there exists a weak solution to our problem (1.15)$_{1-2}$, (1.17), (1.16), (1.2), (1.5), (1.6), (1.8) and (1.9) in the sense of Definition 1 such that

$$\varrho \in L^\infty(\Omega), \qquad \mathbf{u} \in W^{1,q}(\Omega; \mathbb{R}^3), \quad \vartheta \in W^{1,q}(\Omega) \ \text{ for all } 1 \leq q < \infty,$$

and $\varrho \geq 0$, $\vartheta > 0$ a.e. in Ω.

A similar result in two space dimensions can be found in the paper [Pecharová Pokorný 2010], for $\gamma > 2$ and $m = l + 1 > \frac{\gamma - 1}{\gamma - 2}$.

2.2.2 Weak and variational entropy solution

In this section we shall explain the main ideas connected with results from papers [Novotný Pokorný 2011a], [Novotný Pokorný 2011b] and [Jesslé et al. 2014]. The main disadvantage of the results from the previous section ([Mucha Pokorný 2009]) is that the estimate of the velocity gradient is deduced from the momentum equation which means that it depends on the density. The main novelty of the aforementioned series of papers considered in this chapter is that the estimate of the velocity is deduced from the entropy inequality. It is then independent of any other unknown quantities.

We present the following results

Theorem 2 (Dirichlet boundary conditions.) [**Novotný Pokorný 2011a**] *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^3$, $\mathbf{f} \in L^\infty(\Omega; \mathbb{R}^3)$, $\Theta_0 \geq K_0 > 0$ a.e. at $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$. Let $\gamma > \frac{3}{2}$, $m > \max\left\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\right\}$, $l = 0$. Then there exists a variational entropy solution to (1.15)$_{1-2}$, (1.18), (1.3), (1.4), (1.6), (1.8), (1.9) and (1.16) in the sense of Definition 4. Moreover, $\varrho \geq 0$, $\vartheta > 0$ a.e. in Ω and $(\varrho, \mathbf{u})$ is a renormalized solution to the continuity equation in the sense of Definition 6.*

In addition, if $m > 1$ and $\gamma > \frac{5}{3}$, then the solution is a weak solution in the sense of Definition 2.

Theorem 3 (Dirichlet boundary conditions.) [**Novotný Pokorný 2011b**] *Let* Ω *be a* C^2 *be a bounded domain in* $\mathbb{R}^3$, $\mathbf{f} \in L^\infty(\Omega; \mathbb{R}^3)$, $\Theta_0 \geq K_0 > 0$ *a.e. at* $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$. *Let* $\gamma > 1$, $m > \max\left\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\right\}$, $l = 0$. *Then there exists a variational entropy solution to (1.15)*$_{1-2}$*, (1.18), (1.3), (1.4), (1.6), (1.8), (1.9) and (1.16) in the sense of Definition 4. Moreover,* $\varrho \geq 0$, $\vartheta > 0$ *a.e. in* Ω *and* $(\varrho, \mathbf{u})$ *is a renormalized solution to the continuity equation in the sense of Definition 6.*

In addition, if $m > \max\{1, \frac{2\gamma}{3(3\gamma-4)}\}$ *and* $\gamma > \frac{4}{3}$*, then the solution is a weak solution in the sense of Definition 2.*

Theorem 4 (Navier boundary conditions.) [**Jesslé et al 2014**] *Let* Ω *be a* C^2 *bounded domain in* $\mathbb{R}^3$, $\mathbf{f} \in L^\infty(\Omega; \mathbb{R}^3)$, $\Theta_0 \geq K_0 > 0$ *a.e. at* $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$. *Let* $\gamma > 1$, $m > \max\left\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\right\}$, $l = 0$. *Then there exists a variational entropy solution to (1.15)*$_{1-2}$*, (1.18), (1.3), (1.4), (1.6), (1.8), (1.10) and (1.16) in the sense of Definition 5. Moreover,* $\varrho \geq 0$, $\vartheta > 0$ *a.e. in* Ω *and* $(\varrho, \mathbf{u})$ *is a renormalized solution to the continuity in the sense of Definition 6.*

In addition, if $m > 1$ *and* $\gamma > \frac{5}{4}$*, then the solution is a weak solution in the sense of Definition 3.*

Remark 2.1 (i) Note that the results of Theorem 2 hold also for the Navier boundary conditions, just the proof in [Novotný Pokorný 2011a] was performed for the Dirichlet ones.
(ii) In fact, the paper [Novotný Pokorný 2011b] contains a weaker result than Theorem 3. However, as explained in [Mucha et al 2018], to obtain Theorem 3, it is enough to modify slightly at one step the proof for the limit passages.
(iii) It is worth mentioning that the result of Theorem 4 is stronger than the result of Theorem 3 in the sense that the weak solution exists for larger interval of γ.

2.2.3 Two dimensional flow

We consider our system of equations (1.15) with the boundary conditions (1.8)–(1.9) and the given total mass (1.16) in a bounded domain $\Omega \subset \mathbb{R}^2$. We assume the viscous part of the stress tensor in the form (1.3) ($N = 2$) and the heat flux in the form (1.6). Moreover, we take $L = const$ in (1.8). We assume for $\gamma > 1$ the pressure law in the form (1.5) or, formally for $\gamma = 1$, we take

$$p = p(\varrho, \vartheta) = \varrho\vartheta + \frac{\varrho^2}{\varrho + 1} \ln^\alpha(1 + \varrho) \tag{2.31}$$

with $\alpha > 0$. The corresponding specific internal energy fulfils the Maxwell relation (1.12)

$$e = e(\varrho, \vartheta) = \frac{\ln^{\alpha+1}(1 + \varrho)}{\alpha + 1} + c_v\vartheta, \qquad c_v = const > 0,$$

and the specific entropy is

$$s(\varrho, \vartheta) = \ln \frac{\vartheta^{c_v}}{\varrho} + s_0.$$

We consider weak solutions to the problem above defined similarly as in Definition 2 with the corresponding modifications for the pressure law (2.31). This problem was studied in [Novotný Pokorný 2011c] for both (1.5) and (2.31). The improvement for the pressure law (2.31) can be found in the later paper [Pokorný 2011]. The corresponding results read as follows

Theorem 5 (2D flow.) [Novotný Pokorný 2011] & [Pokorný 2011] *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^2$, $\mathbf{f} \in L^\infty(\Omega; \mathbb{R}^2)$, $\Theta_0 \geq K_0 > 0$ a.e. on $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$, $L > 0$.*
(i) Let $\gamma > 1$, $m > 0$. Then there exists a weak solution to our problem with the pressure law (1.5).
(ii) Let $\alpha > 1$ and $\alpha \geq \frac{1}{m}$, $m > 0$. Then there exists a weak solution to our problem with the pressure law (2.31).
Moreover, $(\varrho, \mathbf{u})$ extended by zero outside of Ω is a renormalized solution to the continuity equation.

2.3 Comments to the results, main used techniques

We now present basic ideas of the proofs of the theorems presented above. Generally, the methods are based on standard ideas from the theory of partial differential equations. They consist of the following steps

- formulation of the approximate problem
- solvability of the approximate problem
- estimates of the solutions to the approximate problems

- limit passage to the original problem

The form of the approximate problem is closely connected with so-called *a priori estimates*. These are estimates of solutions to the original problem (even though, at that moment we might not know whether the solution really exists). The advantage of having such estimates is connected with the fact that they provide us ideas what kind of estimates we may hope to get from our approximate problems and they also provide us hint that the approximate problems should be constructed in such a way that we shall be able to extract uniform estimates of such type. Another important issue in the construction of the approximate problem is that we should be able (relatively easily) construct solutions to these problems.

The last step of our procedure which is at least for nonlinear problems the hardest one is to verify that the limit of the sequence of solutions to our approximate problem indeed solves the original problem. This is closely connected, again in case of nonlinear problem, to the problem of *compactness* of the solutions since the estimates ensure us typically only estimates which are not sufficient to pass to the limit in the nonlinear terms. Typical sources of the compactness are compact embedding of some functions spaces (in our case of Sobolev spaces to the Lebesgue spaces which can be applied in our situation to the sequence of velocities and temperatures) and *compensated compactness tools* which allow to prove compactness even in the case when the estimates do not provide any compact embeddings (in our case, this is the situation for the sequence of densities). Let us discuss these issues in more details.

Let us fix, e.g., the weak solutions for the total energy/entropy formulation in the case of the Dirichlet boundary conditions. In this case we can have the following a priori estimates (we will mention how to obtain these estimates for the approximate problems later)

$$\|\mathbf{u}\|_{1,2} + \|\nabla \vartheta^{\frac{m}{2}}\|_2 + \|\nabla \log \vartheta\|_2 + \|\vartheta\|_{L^1(\partial\Omega)} + \|\varrho\|_{\gamma+\Theta} \leq C, \tag{2.32}$$

where $\Theta > 0$ for $\gamma > \frac{3}{2}$ and C depends only on the data of our problem.

2.3.1 Formulation of the approximate problem

In order to formulate our approximate problem, we need to keep in mind that the problem should be solvable and at the same moment, at certain stage, it should provide us with estimates (2.32). Typical tools used in this case are the following

- elliptic regularization
- finite dimensional approximation (or Galerkin approximation)
- reformulation to logarithms of sought quantities
- adding higher powers

We explain how this general idea is applied to solve our problem. We replace the continuity equation $(1.15)_1$ in Ω by

$$-\varepsilon\Delta\varrho + \operatorname{div}(\varrho\mathbf{u}) + \varepsilon\varrho = \varepsilon h \tag{2.33}$$

in Ω with the Neumann boundary condition on $\partial\Omega$

$$\frac{\partial\varrho}{\partial\mathbf{n}} = 0.$$

First of all, for given sufficiently regular velocity the existence and uniqueness of a solution is just a consequence of the Riesz representation theorem. Further, if h is non-negative, then also ϱ is non-negative. This follows by integration (2.33) over the set, where $\varrho < 0$; the fact that we may integrate by parts follows by Sard's theorem. Finally, the total mass is given by the "total mass" of the function h which is just consequence of integrating (2.33) over Ω.

Next, to approximate the momentum equation, we first add $\delta\varrho^\beta$ for $\delta > 0$ and $\beta \gg 1$ to the pressure function and we apply the finite dimensional projection, i.e. we look for Galerkin approximation of the velocity. Then, after suitable linearization, the existence of a solution for given density to this problem is trivial.

Finally, we also regularize the internal energy balance by adding terms with higher power of the temperature ($\kappa_\delta(\vartheta) = \kappa(\vartheta) + \delta\vartheta^B$) and by adding suitable terms in logarithms of the temperature. Then existence of a solution (after linearization) is just a consequence of the standard elliptic theory. The reason of adding the logarithmic terms is the fact that having constructed the logarithm of the temperature we immediately obtain that the temperature must be strictly positive; a maximum principle yielding this information is not evident here.

2.3.2 Solvability of the approximate problem

Due to the fact that each of the (possibly linearized) approximate problems is solvable, to put the equations together the technique of *fixed points* is usually applied. In this case, a version of *the Schauder fixed point theorem* (sometimes called *Schaeffer's fixed point theorem*, see e.g. [Evans 1998]) can be applied. Since the compactness of the corresponding operator is trivial by compact embedding theorems, we only need to verify *a priori estimates* of the possible fixed points in suitable spaces.

They can be obtained by means of the approximate versions of the entropy and total energy balances integrated over Ω. Due to the fact that the temperature is strictly positive and all functions are sufficiently regular, we can deduce both identities directly and we can obtain (except for the density) basically estimates from (2.32). The density can be estimated directly from the approximate continuity equation and higher integrability and differentiability estimates follow by standard elliptic theory. Whence the solvability is proved.

2.3.3 Limit passage to the original problem

We have now three parameters and we need to pass to the limit with $N \to \infty$ (Galerkin approximation for the momentum equation), $\varepsilon \to 0$ (elliptic regularization of the continuity equation) and $\delta \to 0$ (higher powers in the density and temperature), exactly in this order. The first limit passage is not difficult; just to show the strong convergence of the velocity gradient (recall that such a term appears in the internal energy balance) we have to employ the technique of *energy equality* before and after the limit passage in the momentum equation. Since we are not able to repeat this argument for the following limit passages, we have to replace the internal energy balance by the total energy balance and the entropy balance. Due to the sufficient regularity of the density and temperature, this is still possible. The second limit passage, when $\varepsilon \to 0$, already faces the main difficulty of the compressible Navier–Stokes(–Fourier) system: the lack of direct estimates and later also compactness of the sequence of the densities. To solve the first issue, the estimates based on the application of *the Bogovskii operator* are used. More precisely, we apply as test function the solution of the problem

$$\operatorname{div} \boldsymbol{\varphi} = \varrho^{\Theta} - \frac{1}{|\Omega|} \int_{\Omega} \varrho^{\Theta} \, \mathrm{d}x$$

with homogeneous Dirichlet boundary condition. It can be shown that there exist a special branch of solutions (indeed, the problem has infinitely many solutions) such that the solution is unique and, moreover, its L^p-norm of the gradient is controlled by the L^p-norm of the right-hand side. After integrating by parts, the pressure term yields

$$\int_\Omega \big(\varrho^\gamma + \varrho\vartheta + \delta\varrho^\beta\big)\varrho^\Theta \,\mathrm{d}x - \frac{1}{|\Omega|}\int_\Omega \varrho^\Theta \,\mathrm{d}x \int_\Omega \big(p(\varrho,\theta) + \delta\varrho^\beta\big)\,\mathrm{d}x$$
$$\sim \int_\Omega \big(p(\varrho,\theta) + \delta\varrho^\beta\big)\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}x = RHS$$

which, after estimating the terms on the right-hand side (recall also that the L^1-norm of the density is uniformly controlled), provides the sought estimates of the density.

Next problem is the compactness of the density as the previous estimate leads only to the control of the density sequence in some L^p-spaces. To conclude the strong convergence of the density, *compensated compactness tools*, based in particular on the *renormalization of the continuity equation*, must be employed. While in this step only *the effective viscous flux identity* must be combined with *the renormalized continuity equation*, in the next step, when $\delta \to 0$, we also need to show that the limit velocity and density solve the continuity equation in *the renormalized sense*. This can be verified due to an estimate of a quantity called *the oscillation defect measure* which in fact partially controls the lack of convergence of a non-linear quantity due to the weak convergence. All details can be found in the paper [Novotný Pokorný 2011a].

Finally, if γ is close to 1, the estimate coming from the Bogovskii operator is not any more able to help us, since Θ can be positive only for $\gamma > \frac{3}{2}$. Therefore, a different technique, based on special choice of the test function must be employed. Here, it leads to estimates of the type

$$\sup_{x_0\in\overline{\Omega}} \int_\Omega \frac{p(\varrho,\vartheta)}{|x-x_0|^\alpha}\,\mathrm{d}x \le C$$

for some $\alpha > 0$. More details can be found in particular in the overview paper [Kreml et al 2018].

2.3.4 Existence result in two dimensions

The proof in the two-dimensional setting follows more or less the same lines. The only difference, in the case when the pressure has almost linear behaviour with respect to the density (i.e., the form (2.31)), is in some estimates, the Lebesgue spaces must be replaced by suitable *Orlicz spaces*. Otherwise the proof is the same and in a sense, even slightly simpler.

3 Related problems

This section contains results which are connected with more complex problems than just existence of solutions to steady compressible Navier–Stokes–Fourier system, however, which are closely connected with the results from the previous section. It is the time-periodic problem from the compressible Navier–Stokes–Fourier system, where the technique for the steady problem is combined with the technique for the evolutionary one, and two problems, which are more complex and the system from the last sections is only part of the whole system (steady flow with radiation and steady flow of chemically reacting mixtures). The results are described in the following subsections.

3.1 Compressible fluid flow with radiation

We now present the result from paper [Kreml et al. 2013], where a steady flow of a radiative gas has been considered. We are not going into details of its modelling, more information can be found e.g. in aforementioned paper and references therein. We consider the following system of equations in a bounded $\Omega \subset \mathbb{R}^3$

$$\begin{aligned} \operatorname{div}(\varrho\mathbf{u}) &= 0, \\ \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbb{S} + \nabla p &= \varrho\mathbf{f} - \mathbf{s}_F, \\ \operatorname{div}(\varrho E\mathbf{u}) &= \varrho\mathbf{f}\cdot\mathbf{u} - \operatorname{div}(p\mathbf{u}) + \operatorname{div}(\mathbb{S}\mathbf{u}) - \operatorname{div}\mathbf{q} - s_E, \\ \lambda I + \boldsymbol{\omega}\cdot\nabla_x I &= S, \end{aligned} \tag{3.34}$$

where the last equation describes the transport of radiative intensity denoted by I. The right-hand side S is a given function of I, $\boldsymbol{\omega}$ and $\mathbf{u}$. The quantity $\mathbf{s}_F$ denotes the radiative flux and s_E is the radiative energy. The viscous part of the stress tensor is taken in the form (1.3) with the temperature dependent viscosities as in (1.3) such that

$$\mu(\vartheta) \sim (1+\vartheta)^a, \qquad 0 \le \xi(\vartheta) \le C(1+\vartheta)^a$$

for $0 \le a \le 1$. The pressure is considered in the form (1.5) and the heat flux fulfils (1.6), L is a bounded function ($l = 0$). The system is considered together with the homogeneous Dirichlet boundary conditions for the velocity (1.9) and the Newton boundary condition for the heat flux (1.10).

We also prescribe the total mass of the fluid (1.16). The main result reads as follows

Theorem 6 (Steady radiative flow.) [Kreml et al 2013] *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^3$, $\mathbf{f} \in L^\infty(\Omega;\mathbb{R}^3)$, $\Theta_0 \geq K_0 > 0$ a.e. at $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$, $M > 0$. Moreover, let*

$$
\begin{aligned}
&a \in (0,1],\\
&\gamma > \max\Big\{\frac{3}{2}, 1 + \frac{1-a}{6a} + \frac{1}{2}\sqrt{\frac{4(1-a)}{3a} + \frac{(1-a)^2}{9a^2}}\Big\},\\
&m > \max\Big\{1-a, \frac{1+a}{3}, \frac{\gamma(1-a)}{2\gamma-3}, \frac{\gamma(1-a)^2}{3(\gamma-1)^2 a - \gamma(1-a)},\\
&\qquad \frac{1-a}{6(\gamma-1)a-1}, \frac{1+a+\gamma(1-a)}{3(\gamma-1)}\Big\}.
\end{aligned}
$$

Then there exists a variational entropy solution to system (3.34). *Moreover, the pair* $(\varrho, \mathbf{u})$ *is a renormalized solution to the continuity equation.*

If additionally

$$
\begin{aligned}
&\gamma > \max\Big\{\frac{5}{3}, \frac{2+a}{3a}\Big\},\\
&m > \max\Big\{1, \frac{(3\gamma-1)(1-a)}{3\gamma-5}, \frac{(3\gamma-1)(1-a)+2}{3(\gamma-1)},\\
&\qquad \frac{(1-a)(\gamma(2-3a)+a)}{a(6\gamma^2-9\gamma+5)-2\gamma}\Big\},
\end{aligned}
$$

then this solution is a weak solution.

3.2 Time-periodic solution

We describe the problem studied in [Feireisl et al. 2012b]. We consider $(1.1)_{1-2}$ together with (1.11), with the Dirichlet boundary conditions (1.9) for the velocity and the Newton boundary conditions (1.8) for the temperature with $L(\vartheta) = d = const$. The initial conditions (1.7) are replaced by the fact that all functions are time-periodic with the period $T_{per} > 0$. We consider the Fourier law (1.6) and the pressure law (1.4) and its consequences for $\gamma = \frac{5}{3}$, i.e. the monoatomic gas (some extensions were considered in

[Axmann Pokorný 2015]). Note, however, that we must assume in the pressure additionally a radiation term (the term can be justified from physics and, in mathematical treatment, plays an important role), i.e.

$$
\begin{aligned}
p(\varrho,\vartheta) &= p_0(\varrho,\vartheta) + \frac{a}{3}\vartheta^4,\\
e(\varrho,\vartheta) &= e_0(\varrho,\vartheta) + \frac{a}{\varrho}\vartheta^4,\\
s(\varrho,\vartheta) &= s_0(\varrho,\vartheta) + \frac{4a}{3\varrho}\vartheta^3,
\end{aligned}
\tag{3.35}
$$

where p_0, e_0 and s_0 fulfill (1.4), (1.12)–(1.14) with $\gamma = \frac{5}{3}$. We also prescribe the total mass (1.16).

When dealing with time-periodic problems, it is convenient to consider all quantities defined on a time "sphere"

$$
\mathcal{S}^1 = [0, T_{per}]|_{\{0,T_{per}\}}.
$$

Definition 7 (Time-periodic solution.) *We say that a triple $\{\varrho, \mathbf{u}, \vartheta\}$ is a time-periodic weak solution to the Navier–Stokes–Fourier system (1.1)–(1.4), (3.35), (1.6), (1.8), (1.9), (1.11) and (1.16) if the following holds:*

- *the solution belongs to the class $\varrho \geq 0$, $\vartheta > 0$ a.e.,*

$$
\varrho \in L^\infty(\mathcal{S}^1; L^{5/3}(\Omega)),\ \vartheta \in L^\infty(\mathcal{S}^1; L^4(\Omega)),\ \mathbf{u} \in L^2(\mathcal{S}^1; W_0^{1,2}(\Omega;\mathbb{R}^3)),
$$
$$
\vartheta^{3/2}, \ln\vartheta \in L^2(\mathcal{S}^1; W^{1,2}(\Omega))
$$

- *equation of continuity (1.1)$_1$ is satisfied in the sense of renormalized solutions,*

$$
\int_{\mathcal{S}^1}\int_\Omega \big(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla\varphi + (b(\varrho) - b'(\varrho)\varrho)\operatorname{div}\mathbf{u}\varphi\big)\ \mathrm{d}x\ \mathrm{d}t = 0
$$

  for any $b \in C^\infty[0,\infty)$, $b' \in C_c^\infty[0,\infty)$, and any test function $\varphi \in C^\infty(\mathcal{S}^1 \times \overline{\Omega})$

- *momentum equation (1.1)$_2$ holds in the sense of distributions:*

$$
\begin{aligned}
\int_{\mathcal{S}^1}\int_\Omega \big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} + p(\varrho,\vartheta)\operatorname{div}\boldsymbol{\varphi}\big)\ \mathrm{d}x\ \mathrm{d}t\\
= \int_{\mathcal{S}^1}\int_\Omega \big(\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta) : \nabla\boldsymbol{\varphi} - \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\big)\ \mathrm{d}x\ \mathrm{d}t
\end{aligned}
\tag{3.36}
$$

  for any $\boldsymbol{\varphi} \in C_c^\infty(\mathcal{S}^1 \times \Omega; \mathbb{R}^3)$

- *entropy equation (1.11) with the boundary condition (1.8) are satisfied in the sense of the integral identity*

$$\int_{\mathcal{S}^1}\int_\Omega \Big(\varrho s(\varrho,\vartheta)\partial_t\psi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi + \frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\cdot\nabla\psi\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$= \int_{\mathcal{S}^1}\int_{\partial\Omega}\frac{d}{\vartheta}(\vartheta-\Theta_0)\psi\ \mathrm{d}S\ \mathrm{d}t - <\sigma;\psi> \tag{3.37}$$

for any $\psi \in C^\infty(\mathcal{S}^1\times\overline{\Omega})$*, where* $\sigma \in \mathcal{M}^+(\mathcal{S}^1\times\overline{\Omega})$ *is a non-negative measure satisfying*

$$\sigma \geq \frac{1}{\vartheta}\Big(\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta) : \nabla\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\Big) \tag{3.38}$$

- *the total energy balance*

$$\begin{aligned}&\int_{\mathcal{S}^1}\Big(\partial_t\psi\int_\Omega\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\Big)\ \mathrm{d}x\Big)\ \mathrm{d}t\\ =&\int_{\mathcal{S}^1}\psi\Big(\int_{\partial\Omega} d(\vartheta-\Theta_0)\ \mathrm{d}S - \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x\Big)\ \mathrm{d}t\end{aligned} \tag{3.39}$$

  holds for any $\psi \in C^\infty(\mathcal{S}^1)$.

It is not difficult to see that the entropy production inequality (3.37) reduces to (1.11) as soon as the solution is smooth enough. In the paper [Feireisl et al. 2012b], the following result was proved:

Theorem 7 (Time-periodic solution.) [Feireisl et al 2012c] *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with a boundary of class* $C^{2+\nu}$*. Suppose that the thermodynamic functions* p*,* e*, and* s *satisfy hypotheses (3.35), (1.4) and (1.12)–(1.14). Let* $\mathbf{f} \in L^\infty(\mathcal{S}^1\times\Omega;\mathbb{R}^3)$*.*

Then for any $M > 0$ *the Navier–Stokes–Fourier system possesses at least one time-periodic-solution* $\{\varrho,\mathbf{u},\vartheta\}$ *in the sense specified above such that*

$$\int_\Omega \varrho(t,\cdot)\ \mathrm{d}x = M \quad \textit{for all } t \in \mathcal{S}^1.$$

3.3 Mathematical theory for steady multicomponent flow

We consider the system of equations

$$
\begin{aligned}
\operatorname{div}(\varrho \mathbf{u}) &= 0,\\
\operatorname{div}(\varrho \mathbf{u}\otimes \mathbf{u}) - \operatorname{div}\mathbb{S} + \nabla p &= \varrho \mathbf{f},\\
\operatorname{div}(\varrho E\mathbf{u}) + \operatorname{div}(p\mathbf{u}) + \operatorname{div}\mathbf{Q} - \operatorname{div}(\mathbb{S}\mathbf{u}) &= \varrho \mathbf{f}\cdot \mathbf{u},\\
\operatorname{div}(\varrho Y_k \mathbf{u}) + \operatorname{div}\mathbf{F}_k &= \omega_k, \quad k \in \{1,\dots,L\}
\end{aligned}
\tag{3.40}
$$

with the boundary conditions

$$
\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \mathbf{F}_k \cdot \mathbf{n}|_{\partial\Omega} \quad = 0, -\mathbf{Q}\cdot\mathbf{n} + L(\vartheta - \vartheta_0) = 0. \tag{3.41}
$$

The total mass of the mixture is prescribed,

$$
\int_\Omega \varrho \,\mathrm{d}x = M > 0. \tag{3.42}
$$

The meaning of most quantities (the density of the mixture ϱ, its temperature ϑ and velocity $\mathbf{u}$) is the same as in the case of single constituted fluid. Note just that $\mathbf{u}$ is the barycentric velocity and we assume that it is enough to model the flow of the whole mixture just by this velocity. Moreover, we also do not distinguish between the temperature of each constituent. On the other hand, we distinguish the amount of each constituent at each place by looking for the mass fractions $Y_k := \frac{\varrho_k}{\varrho}$, where $\sum_{k=1}^L \varrho_k = \varrho$, hence $\sum_{k=1}^L Y_k = 1$. We assume this even though we cannot exclude that the density $\varrho = 0$ at some places. The stress tensor $\mathbb{S}$ has the same from as in the previous sections, i.e. it is given by (1.3). As we have to assume that the molar masses are the same (due to mathematical reasons), the pressure p is given by (1.6). The specific total energy $E = \frac{1}{2}|\mathbf{u}|^2 + e$, where the specific internal energy $e = \frac{\varrho^{\gamma-1}}{\gamma-1} + \vartheta \sum_{k=1}^L c_{vk} Y_k$ with c_{vk}, the constant-volume specific heat coefficients. Then $c_{pk} = c_{vk} + 1$, where c_{pk} is the constant-pressure specific heat coefficients. Next, the heat flux $\mathbf{Q} = \mathbf{q} + \sum_{k=1}^L c_{pk}\vartheta \mathbf{F}_k$ with $\mathbf{q}$ the part given by the Fourier law (1.6).

The multicomponent fluxes $\mathbf{F}_k$ have to fulfill $\sum_{k=1}^L \mathbf{F}_k = \mathbf{0}$ and they are given by

$$
\mathbf{F}_k = -Y_k \sum_{l=1}^L D_{kl} \nabla Y_l,
$$

where $\mathbb{D} = \{D_{kl}\}_{k,l=1}^{L}$ is the multicomponent diffusion matrix. Except for the structural assumption we take

$$|D_{ij}(\vartheta, \vec{Y})| \leq C(|\vec{Y}|)(1 + \vartheta^b)$$

for some $b > 0$ and. Finally, the species production rates ω_k fulfill $\sum_{k=1}^{L} \omega_k = 0$ as well as

$$-\sum_{k=1}^{L} g_k \omega_k \geq 0,$$

where g_k are the Gibbs functions. More details can be found in the paper [Piasecki Pokorný 2017] and [Piasecki Pokorný 2018] and in particular also in the monograph [Giovangigli 1999].

First, we define the objects we want to construct.

Definition 8 (Multicomponent flow; weak solution.) *We say the set of functions $(\varrho, \mathbf{u}, \vartheta, \vec{Y})$ is a weak solution to problem (3.40)–(3.42) with assumptions stated above, provided*

- $\varrho \geq 0$ *a.e. in* Ω, $\varrho \in L^{6\gamma/5}(\Omega)$, $\int_\Omega \varrho \,\mathrm{d}x = M$
- $\mathbf{u} \in W_0^{1,2}(\Omega)$, $\varrho|\mathbf{u}|$ *and* $\varrho|\mathbf{u}|^2 \in L^{\frac{6}{5}}(\Omega)$
- $\vartheta \in W^{1,2}(\Omega) \cap L^{3m}(\Omega)$, $\varrho\vartheta, \varrho\vartheta|\mathbf{u}|, \mathbb{S}\mathbf{u}, \kappa|\nabla\vartheta| \in L^1(\Omega)$
- $\vec{Y} \in W^{1,2}(\Omega)$, $Y_k \geq 0$ *a.e. in* Ω, $\sum_{k=1}^{L} Y_k = 1$ *a.e. in* Ω, $\mathbf{F}_k \cdot \mathbf{n}|_{\partial\Omega} = 0$

and the following integral equalities hold
• *the weak formulation of the continuity equation*

$$\int_\Omega \varrho\mathbf{u} \cdot \nabla\psi \,\mathrm{d}x = 0 \tag{3.43}$$

holds for any test function $\psi \in C^\infty(\overline{\Omega})$
• *the weak formulation of the momentum equation*

$$-\int_\Omega \big(\varrho\,(\mathbf{u} \otimes \mathbf{u}) : \nabla\boldsymbol{\varphi} - \mathbb{S} : \nabla\boldsymbol{\varphi}\big) \,\mathrm{d}x - \int_\Omega \pi \operatorname{div}\boldsymbol{\varphi} \,\mathrm{d}x = \int_\Omega \varrho\mathbf{f} \cdot \boldsymbol{\varphi} \,\mathrm{d}x \tag{3.44}$$

holds for any test function $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$
• *the weak formulation of the species equations*

$$-\int_\Omega Y_k \varrho\mathbf{u} \cdot \nabla\psi \,\mathrm{d}x - \int_\Omega \mathbf{F}_k \cdot \nabla\psi \,\mathrm{d}x = \int_\Omega \omega_k \psi \,\mathrm{d}x \tag{3.45}$$

holds for any test function $\psi \in C^\infty(\overline{\Omega})$ *and for all* $k = 1, \dots, L$

- *the weak formulation of the total energy balance*

$$
\begin{aligned}
&-\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right) \mathbf{u} \cdot \nabla\psi \ \mathrm{d}x + \int_\Omega \kappa \nabla\vartheta \cdot \nabla\psi \ \mathrm{d}x \\
&-\int_\Omega \left(\sum_{k=1}^L h_k \mathbf{F}_k\right) \cdot \nabla\psi \ \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u}\psi \ \mathrm{d}x - \int_\Omega (\mathbb{S}\mathbf{u}) \cdot \nabla\psi \ \mathrm{d}x \qquad (3.46)\\
&+\int_\Omega \pi \mathbf{u} \cdot \nabla\psi \ \mathrm{d}x - \int_{\partial\Omega} L(\vartheta - \vartheta_0)\psi \, \mathrm{d}S
\end{aligned}
$$

holds for any test function $\psi \in C^\infty(\overline{\Omega})$.

The admissible range of γ in the pressure law for which we are able to show existence of weak solutions in the above sense is limited mostly by the terms $\varrho|\mathbf{u}|^2\mathbf{u}$ and $\mathbb{S}\mathbf{u}$ in the weak formulation of total energy balance. Therefore, similarly as in the single component flow, we replace the total energy balance $(3.40)_3$ by the entropy inequality specified in Definition 8 below. Note also that for the Navier boundary conditions for the velocity it is possible to obtain the existence of both weak and variational entropy solutions (see below) under less restrictive assumptions on γ, cf. [Piasecki Pokorný 2018].

Definition 9 (Multicomponent flow; variational entropy solution.) *We say the set of functions* $(\varrho, \mathbf{u}, \vartheta, \vec{Y})$ *is a variational entropy solution to problem (3.40–3.42) with assumptions stated above, provided*

- $\varrho \geq 0$ *a.e. in* Ω, $\varrho \in L^{s\gamma}(\Omega)$ *for some* $s > 1$, $\int_\Omega \varrho \ \mathrm{d}x = M$
- $\mathbf{u} \in W_0^{1,2}(\Omega)$, $\varrho\mathbf{u} \in L^{\frac{6}{5}}(\Omega)$
- $\vartheta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega)$, $r > 1$, $\varrho\vartheta, \mathbb{S} : \frac{\nabla\mathbf{u}}{\vartheta}, \kappa\frac{|\nabla\vartheta|^2}{\vartheta^2}, \kappa\frac{\nabla\vartheta}{\vartheta} \in L^1(\Omega)$, $\frac{1}{\vartheta} \in L^1(\partial\Omega)$
- $\vec{Y} \in W^{1,2}(\Omega)$, $Y_k \geq 0$ *a.e. in* Ω, $\sum_{k=1}^L Y_k = 1$ *a.e. in* Ω, $\mathbf{F}_k \cdot \mathbf{n}|_{\partial\Omega} = 0$

satisfy equations (3.43)–(3.45), *the following entropy inequality*

$$
\begin{aligned}
&\int_\Omega \frac{\mathbb{S}:\nabla \mathbf{u}}{\vartheta}\psi \,\mathrm{d}x + \int \kappa \frac{|\nabla\vartheta|^2}{\vartheta^2}\psi \,\mathrm{d}x - \int_\Omega \sum_{k=1}^L \omega_k (c_{pk} - c_{vk}\log\vartheta + \log Y_k)\psi \,\mathrm{d}x \\
&\quad + \int_\Omega \psi \sum_{k,l=1}^n D_{kl}\nabla Y_k \cdot \nabla Y_l \,\mathrm{d}x + \int_{\partial\Omega} \frac{L}{\vartheta}\vartheta_0 \psi \,\mathrm{d}S \le \int \frac{\kappa \nabla\vartheta\cdot\nabla\psi}{\vartheta}\,\mathrm{d}x \\
&\qquad - \int_\Omega \varrho s \mathbf{u}\cdot\nabla\psi \,\mathrm{d}x - \int_\Omega \log\vartheta \Big(\sum_{k=1}^L \mathbf{F}_k c_{vk}\Big)\cdot\nabla\psi \,\mathrm{d}x \\
&\qquad\quad + \int_\Omega \Big(\sum_{k=1}^L \mathbf{F}_k \log Y_k\Big)\cdot\nabla\psi \,\mathrm{d}x + \int_{\partial\Omega} L\psi \,\mathrm{d}S \qquad (3.47)
\end{aligned}
$$

for all non-negative $\psi \in C^\infty(\overline{\Omega})$ *and the global total energy balance (i.e.* (3.46) *with* $\psi \equiv 1$*)*

$$\int_{\partial\Omega} L(\vartheta - \vartheta_0)\,\mathrm{d}S = \int_\Omega \varrho \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x. \qquad (3.48)$$

Note, however, that (3.47) does not contain all terms from the formally deduced entropy identity, some of them are missing. These terms are formally equal to zero due to assumptions that ω_k and $\mathbf{F}_k$ sum up to zero. We removed them from the formulation of the entropy inequality due to the fact that we cannot exclude the situation that $\varrho = 0$ in some large portions of Ω (with positive Lebesgue measure), thus $\log\varrho$ is not well defined there. However, the variational entropy solution has still the property that any sufficiently smooth variational entropy solution in the sense above is a classical solution to our problem, provided the density is strictly positive in Ω.

We are now in position to formulate our main result.

Theorem 8 (Multicomponent flow.) [Piasecki Pokorný 2017] *Let* $\gamma > 1$, $M > 0$, $m > \max\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\}$, $b < \frac{3m}{2}$. *Let* $\Omega \in C^2$. *Then there exists at least one variational entropy solution to our problem above. Moreover,* $(\varrho, \mathbf{u})$ *is the renormalized solution to the continuity equation.*

In addition, if $m > \max\{1, \frac{2\gamma}{3(3\gamma-4)}\}$, $\gamma > \frac{4}{3}$, $b < \frac{3m-2}{2}$, *then the solution is a weak solution in the sense above.*

Resumé

The presented thesis contains mostly the existence results for equations describing steady flow of heat conducting compressible viscous Newtonian fluid, i.e. for the steady compressible Navier–Stokes–Fourier system under different boundary conditions. It deals with existence of solutions for large data, i.e., we do not try to construct solutions which are close to some known regular solutions. This fact leads to the necessity of considering the weak solutions and their generalizations instead of the classical or strong ones.

The formulation of the problem as a system of balance laws allows several formulations which are equivalent on the level of classical or strong solutions: the balance of mass (the continuity equation) and the balance of the linear momentum can be combined with the internal energy balance, total energy balance or the entropy balance. These three possible formulations are not any more equivalent on the level of weak solutions. It is, however, important to recall that all three types of solutions possess the property of the weak-strong compatibility. In the thesis, it is demonstrated that in different situations (properties of viscosities, different values of physical constants and different boundary conditions for the velocity) existence of solutions for different formulations can be obtained.

Based on similar situation in the evolutionary problems, it is demonstrated that the entropy inequality is an extremely effective tool in this type of problems. It provides useful estimates which are stronger than estimates coming from the energy inequality, and, in addition, the solution based on the entropy inequality (together with a partial information from the total energy balance) exists for the largest set of parameters (the value of the adiabatic constant and the speed growth of the heat conductivity with respect to the temperature).

This observation, together with tools used in the mathematical fluid mechanics and thermodynamics for evolutionary problems (density estimates based on the Bogovskii operator, effective viscous flux identity, renormalized solution to the continuity equation and oscillation defect measure estimates) and tools specific for steady problems (potential estimates of the density up to the boundary, possibility to use total energy balance in the weak formulation) enabled to understand relatively well the problems of existence of

solutions for steady systems describing flow of heat conducting compressible Newtonian fluid.

This technique also helped to study closely related problems like existence of time periodic solutions for heat conducting compressible fluids with physically realistic parameters (including at least the monatomic gas model) or obtain results for more complex systems as chemically reacting gaseous mixtures or flow of gases with radiation. The thesis also includes a very specific result dealing with formulation of the problem with the internal energy balance which was actually the first real large data existence result for steady equations of compressible heat conducting fluids. All the presented results inspired other scientists who used the therein developed tools to study similar problems.

The thesis is divided into two parts. In the first, introductory one, after a short description of the studied problems, the known existence results are formulated in dependence on the parameters of the problem. Furthermore, the main ideas of the existence proofs as well as the necessary tools used therein are briefly explained. Due to the complexity of the problem, all the proofs are long and technically complicated. The second part then contains eight selected most important papers from the perspective of the author of the thesis. They were mostly published in high-ranked journals from the field of partial differential equations or mathematical fluid mechanics and were obtained in collaboration with different leading experts in the field of mathematical fluid mechanics and thermodynamics.

Indeed, especially in the mathematical theory for models of complex fluids, many important questions and problems remained unsolved or even untouched. Dealing with them can bring development of new tools and techniques which may lead to improvement of results for the "simpler" problems, but for sure, will also open new perspective and enable to study problems which, nowadays, we even do not dare to dream about.

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