Perfect mappings and their applications in the theory of compact convex sets

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Summary

We study an interplay between descriptive set theory and theory of compact convex sets. Theory of compact convex sets serves as a general framework for an investigation of Banach spaces as well as more general objects as subsets of Banach spaces, sets of measures etc. Descriptive set theory provides a way how to measure a complexity of given objects. Connections between these two different mathematical disciplines provide a useful insight in both of them.

We recall that theory of Banach spaces is in a way subsumed by theory of compact convex sets via the following procedure. If E is a Banach space, its dual unit ball B_{E^*} endowed with the weak* topology is a compact convex set and E can be viewed as an isometric subspace of the space of all affine continuous functions on B_{E^*}. Further, B_{E^*} is a natural example of a compact topological space. Hence it possesses a rich Borel structure, i.e., it carries the σ–algebra of all Borel sets in B_{E^*}. Borel sets can be more finely distinguished into Borel classes and thus we may ask what is the class of a given object in B_{E^*}.

The main focus of the research presented in the thesis is the application of (nonmetrizable) descriptive set theory in theory of compact convex sets. First we build a theory of Borel classes in topological spaces and show their stability with respect to perfect mappings. This property turns out to be of utmost importance for the applications in the second chapter of the thesis. Since the notion of affine Baire-1 functions is studied in Section 3.1, the second part of Chapter 2 deals with the question of extending Baire-1 functions from subsets of topological spaces. Last but not least, the third part provides a general theory of Borel classes in topological spaces.

The second part of the thesis applies the results of the first part in theory of compact convex sets and Banach spaces. First we solve the abstract Dirichlet problem for Baire-1 functions on compact convex sets, then we investigate the possibility of transferring descriptive properties of strongly affine functions from the set of extreme points.

Resumé

V disertaci studujeme vztahy mezi deskriptivní teorií množin a teorií kompaktních konvexních množin. Teorii kompaktních konvexních množin lze pokládat za obecný rámec, ve kterém je možné kromě Banachových prostorů zkoumat i jejich podmnožiny či prostory měr. Deskriptivní teorie množin poskytuje metodu, jak měřit složitost uvažovaných objektů. Vztahy mezi těmito dvěma matematickými disciplínami poskytují zajímavý vhled do obou z nich.

Připomeňme, že teorie Banachových prostorů je částečně zahrnuta v teorii kompaktních konvexních množin pomocí následující úvahy. Je-li E Banachův prostor, je jeho duální jednotková koule B_{E^*} konvexní množina, jež je kompaktní ve weak* topologii. Dále, E je isometricky vnořen do prostoru spojitých afinních funkcí na B_{E^*}. Jelikož je B_{E^*} kompaktní prostor, lze na něm uvažovat borelovskou strukturu, tj. σ-algebru borelovských množin. Ty lze dále jemněji roztřídit do borelovských tříd, což umožňuje zkoumat borelovskou třídu daného objektu.

Disertační práce je zaměřena na aplikace deskriptivní teorie množin v teorii kompaktních konvexních množin. Po vybudování teorie borelovských tříd v obecných topologických prostorech jsou prezentovány výsledky o jejich stabilitě vzhledem k perfektním zobrazením. Tato vlastnost je klíčová pro pozdější aplikace ve druhé kapitole práce. Jelikož je druhá část práce mimo jiné věnována afinním funkcím první třídy, zabýváme se v Sekci 2.2. rozšiřováním funkcí první třídy z podmnožin topologických prostorů. Závěr první kapitoly je pak věnován obecné teorii borelovských tříd v topologických prostorech.

Druhá část práce aplikuje výsledky první v teorii kompaktních konvexních množin a v Banachových prostorech. Nejprve se věnujeme řešení abstraktní Dirichletovy úlohy na kompaktních konvexních množinách. Druhá sekce Kapitoly 2 zkoumá přenášení deskriptivních vlastností silně afinních funkcí z množiny extremálních bodů.

1. Introduction

A *Banach space* is a real normed linear space which is complete in the metric induced by the norm. In particular, $\mathbb{R}^n$ or $\mathbb{C}^n$ is a Banach space when equipped with the Euclidean norm. The sequence spaces ℓ_p (for $p \in [1, \infty]$), the space c_0 of sequences converging to 0, Lebesgue function spaces $L_p([0,1])$ (for $p \in [1, \infty]$) or the space $C([0,1])$ of continuous functions on $[0,1]$ are classical examples of infinite dimensional Banach spaces.

Banach spaces admit several structures including algebraical, geometrical and topological ones. One can view them as linear spaces, metric spaces or topological spaces. It is also possible to study the interplay of these points of view. There are several natural topologies on a Banach space. The first one is the *norm topology*, induced by the metric generated by the norm. Another very important one is the *weak topology*, which is the weakest topology having the same continuous linear functionals as the norm topology. On a dual space there is another topology – namely the topology of pointwise convergence, which is called the *weak* topology*.

A *compact space* is a topological space K such that each cover of K by open sets admits a finite subcover. For example, the unit interval $[0,1]$ is compact. More generally, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. A topological space K is Lindelöf if any open cover of K admits a countable subcover. A metrizable topological space is Lindelöf if and only if it is separable.

Compact spaces are closely related to Banach spaces. The first result of this kind says that the closed unit ball B_E of a Banach space E is compact (in the norm topology) if and only if the space E has finite dimension. A deeper result is the Banach-Alaoglu theorem saying that the unit ball B_{E^*} of the dual space X^* is compact in the weak* topology for any Banach space E. For a compact space K, let $\mathcal{C}(K)$ stand for the Banach space of all continuous functions on K. Then we can embed any Banach space E to the space $\mathcal{C}(B_{E^*})$ via the canonical embedding, namely, for $x \in E$ we define $\widehat{x}(x^*) = x^*(x)$, $x^* \in B_{E^*}$. The mapping $x \mapsto \widehat{x}$ is then an isometric embedding of E into the space of all continuous affine functions on B_{E^*}.

This example leads to the study of a more general framework, namely to the theory of compact convex sets. Let X be a compact convex subset of a locally convex space E and $\mathfrak{A}^c(X)$ denote the space of all continuous affine functions on X. If $X = B_{E^*}$ for a Banach space E, X is a subset of a locally convex space E^* endowed with the weak* topology and E is identified with the space of all continuous affine functions vanishing at 0. Thus the study of spaces of affine continuous functions on compact convex set can be regarded as a general framework for an investigation of Banach spaces.

When dealing with a topological space K, we want to work with easily definable sets – descriptive ones. A natural descriptive object is the σ-algebra of all *Borel* subsets of K, i.e., the σ-algebra generated by the family of open subsets of K. By their finer distinguishing we can talk about sets of Borel class α for a countable ordinal α. These classes in a way describes complexity of the involved sets. One of their interesting features is their stability with respect to *perfect* mappings. (Examples of perfect mappings are continuous mappings between compact topological spaces.)

The main idea of the thesis is an interplay between theory of compact convex sets and descriptive properties of the involved affine functions, in particular we focus on applications of perfect mappings in theory of compact convex sets.

2. Summary of Chapter 2

We always consider our topological spaces to be Tychonoff, i.e., completely regular. Also, for the sake of simplicity we consider vector spaces to be real.

2.1. **Summary of Section 2.1: Perfect images of absolute Souslin and absolute Borel Tychonoff spaces.** Let us recall classical results on Borel classes and functions in separable metrizable space due to Kuratowski, Hausdorff, etc.

We start with several definitions. If X is a set and $\mathcal{F}$ is a family of subsets in X, then $\mathcal{F}$ is a *sublattice*, if $\emptyset, X \in \mathcal{F}$ and $\mathcal{F}$ is closed with respect to finite unions and intersections. The family $\mathcal{F}$ is an *algebra* if $\mathcal{F}$ is a sublattice that is closed with respect to complements. If $\mathcal{F}$ is a family of sets in a set X, we write $\mathcal{F}_\sigma$ (respectively $\mathcal{F}_\delta$) for all countable unions (respectively intersections) of sets from $\mathcal{F}$. We write χ_A for the characteristic function of a set A and $f|_A$ for the restriction of a function f on A. If $f : X \to Y$ is a mapping from X to a topological space Y, we say that f is $\mathcal{F}$*–measurable*, if $f^{-1}(U) \in \mathcal{F}$ for each open $U \subset Y$.

If $\mathcal{F}$ is a family of sets in a set X, we define abstract Borel classes generated by $\mathcal{F}$ as follows: Let $\Sigma_1(\mathcal{F}) = \mathcal{F}$, $\Pi_1(\mathcal{F}) = \{X \setminus F : F \in \mathcal{F}\}$, and for $\alpha \in (1, \omega_1)$, let

$$\Sigma_\alpha(\mathcal{F}) = \Big(\bigcup_{\beta<\alpha} \Pi_\beta(\mathcal{F})\Big)_\sigma$$

and

$$\Pi_\alpha(\mathcal{F}) = \Big(\bigcup_{\beta<\alpha} \Sigma_\beta(\mathcal{F})\Big)_\delta.$$

The family $\Sigma_\alpha(\mathcal{F})$ is termed the *sets of additive class* α, the family $\Pi_\alpha(\mathcal{F})$ is called the *sets of multiplicative class* α. The sets in $\Delta_\alpha(\mathcal{F}) = \Sigma_\alpha(\mathcal{F}) \cap \Pi_\alpha(\mathcal{F})$ are the *sets of ambiguous class* α.

Further we define the inductive classes of mappings. If Φ is a family of mappings from a set X to a topological space Y, inductively we define Baire classes generated by Φ as follows: Let $\Phi_0 = \Phi$ and for each countable ordinal $\alpha \in (0, \omega_1)$, let Φ_α be the family of all pointwise limits of sequences from $\bigcup_{\beta<\alpha} \Phi_\beta$.

It will be sometimes convenient to denote the starting family of the inductive definition as Φ_1. More precisely, we start from a family denoted as Φ_1 and then Φ_α consists of all pointwise limits of sequences from $\bigcup_{1\leq\beta<\alpha} \Phi_\beta$, $\alpha \in (1, \omega_1)$. The purpose of this convention is that we want to start the generation of mappings between topological spaces from "Baire–one" mappings.

Let $\mathcal{F}$ be an algebra of sets in a set X, Y be a separable metrizable space and let Φ_1 stand for the family of all $\Sigma_2(\mathcal{F})$-measurable mappings from X to Y. Then we get the following analogue of the Lebesgue-Hausdorff-Banach characterization as follows:

A mapping $f\colon X \to Y$ *is* $\Sigma_{\alpha+1}(\mathcal{F})$*-measurable if and only if* $f \in \Phi_\alpha$.

If $\mathcal{F}$ is a metrizable space and $\mathcal{F}$ is the algebra of sets which are both F_σ and G_δ, then the resulting classes are the classical classes of Borel sets (see [19, Section 11.B]). Also, the Lebesgue-Hausdorff-Banach characterization is then a classical result.

If X is a Tychonoff topological space, we write $\mathcal{G}(X)$ for the sublattice of all open subsets of X. A subset A of X is called a *zero set* if $A =$

$f^{-1}(\{0\})$ for a continuous real-valued function f on X. It is clear that such a function f can be chosen with values in $[0,1]$. A *cozero set* is the complement of a zero set. It is easy to check that zero sets are preserved by finite unions and countable intersections. Hence cozero sets are preserved by finite intersections and countable unions. (Zero and cozero sets are termed *functionally closed* and *functionally open* sets in [6, p. 42]).

We recall that *Borel sets* are members of the σ–algebra generated by the family of all open subset of X and *Baire sets* are members of the σ–algebra generated by the family of all cozero sets in X. We recall that a subset A of a topological space X is F_σ, if A can be written as a countable union of closed sets. The complement of an F_σ set is called a G_δ set.

The space X is called *scattered* if its each subset has an isolated point, that is, for each $A \subset X$ there exists $x \in A$ and an open set U such that $A \cap U = \{x\}$. The space X is *σ–scattered*, if X can be written as a countable union of scattered subspaces.

We consider the following families of subsets of X.

(a) The algebra $\mathrm{Bas}(X)$ generated by zero sets. Then

$$\mathrm{Bas}(X) = \{\bigcup_{i=1}^{n}(F_i \setminus H_i) : F_i, H_i \text{ are zero sets in } X, n \in \mathbb{N}\}.$$

(b) The algebra $\mathrm{Bos}(X)$ generated by closed subsets of X. As above,

$$\mathrm{Bos}(X) = \{\bigcup_{i=1}^{n}(F_i \setminus H_i) : F_i, H_i \text{ are closed in } X, n \in \mathbb{N}\}.$$

(c) The algebra $\mathrm{Hs}(X)$ of all *H–sets* (or *resolvable sets*). H–sets are defined in [21, §12, II], where their basic properties are described (see also [20, p. 218]). Let us recall some equivalent definitions. A subset A of a topological space X is an H–set if for any nonempty $B \subset X$ there is a nonempty relatively open $U \subset B$ such that either $U \subset A$ or $U \cap A = \emptyset$. It is clear that H–sets form an algebra containing all open sets. Further, A is an H–set in X if and only if A is the union of a scattered family of sets of the form $F \cap G$ with F closed and G open. (We recall that a family $\mathcal{U}$ of subsets of a topological space is *scattered* if it is disjoint and for each nonempty $\mathcal{V} \subset \mathcal{U}$ there is some $V \in \mathcal{V}$ relatively open in $\bigcup \mathcal{V}$. Thus it follows that a topological space X is scattered if $\{\{x\} : x \in X\}$ is a scattered family.)

For each algebra of sets listed in (a)–(c) we consider the classes of sets defined in Section 1.1.2. For $\alpha \in (1, \omega_1)$, *the sets in* $\Sigma_\alpha(\mathrm{Bos}(X))$ *or* $\Sigma_\alpha(\mathrm{Bas}(X))$ *will be called the sets of additive Borel or Baire class* α, *respectively. Similarly we label the sets in* $\Pi_\alpha(\mathrm{Bos}(X))$ *or* $\Pi_\alpha(\mathrm{Bas}(X))$ *as the sets of multiplicative Borel or Baire class* α, *respectively.*

(d) If we start the Borel hierarchy from the sublattice $\mathcal{G}(X)$ of all open subsets of X, for metrizable spaces we get the standard Borel hierarchy as defined in [19, Section 11.B]. We write $\Sigma^0_\alpha(\mathcal{G}(X))$ and $\Pi^0_\alpha(\mathcal{G}(X))$ for the families obtained by this procedure. We show below its relation to the families defined in (a)–(c). We just mention that a set A belongs to $\Sigma^0_2(\mathcal{G}(X))$ if and only if A is of type F_σ.

In general, $\mathrm{Hs}(X)$ may contain a non-Borel set, in fact $\mathrm{Hs}(X)$ may be a strictly larger family than the system of all Borel sets in X. An easy example is provided by a suitable scattered compact space X. Namely, in this case any subset of X is an H–set, since $\{\{x\} : x \in X\}$ is a scattered family consisting of closed sets. If $X = [0, \omega_1]$ with the order topology and $A \subset [0, \omega_1)$ is a stationary subset, so that $[0, \omega_1) \setminus A$ is also stationary (see [15, Lemma 7.6]), then A is a resolvable non-Borel set (see [29, Lemma 1], [10, p. 296] or [12, Example 4.4]).

We can consider even more general descriptive classes of sets. Let $\mathbb{N}^\mathbb{N}$ denote the space of all sequences of natural numbers. For $\sigma \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$, we write $\sigma|_n$ for the finite sequence $(\sigma(1), \dots, \sigma(n))$. If X is a set and $\mathcal{F}$ is a family of its subsets, we say that $A \subset X$ is a result of the *Souslin operation* applied to the sets $\mathcal{F}$, if

$$A = \bigcup_{\sigma \in \mathbb{N}^\mathbb{N}} \bigcap_{n=1}^{\infty} F_{\sigma|_n},$$

where $F_s \in \mathcal{F}$ for each finite sequence s of natural numbers. We write $\mathcal{S}(\mathcal{F})$ for the family of sets $A \subset X$ obtainable by this procedure.

For families $\mathrm{Bas}(X)$, $\mathrm{Bos}(X)$ and $\mathrm{Hs}(X)$ we consider sets obtainable by the Souslin operation applied to these families.

The main result now can be stated as follows.

Theorem 1. *Let $\varphi\colon X \to Y$ be a continuous surjection of a compact space X onto a compact space Y. Let $\mathcal{F}$ be any of the descriptive class mentioned above and let $B \subset Y$. Then $B \in \mathcal{F}$ if and only if $\varphi^{-1}(B) \in \mathcal{F}$.*

The basic ingredient of the proof is the following selection result.

We recall that a set-valued mapping $F : X \to Y$ is a *usco* mapping if F has compact values and for any closed set $H \subset Y$ the set

$$F^{-1}(H) = \{x \in X : F(x) \cap H \neq \emptyset\}$$

is closed in X.

Lemma 1. *Let X and Y be Hausdorff topological spaces and F be a usco mapping of Y to X with nonempty values. Suppose further that H_n, $n \in \mathbb{N}$, are resolvable sets.*

Then there is a set-valued mapping S of Y to X such that

(a) *$S(y) \subset F(y)$ is a nonempty compact subset of X for every $y \in Y$,*
(b) *$S^{-1}(H_n) \cap S^{-1}(X \setminus H_n) = \emptyset$ for every $n \in \mathbb{N}$, and*
(c) *$S^{-1}(H_n)$ is resolvable in Y for every $n \in \mathbb{N}$.*

For the algebra Bos we can formulate a more general result.

Lemma 2. *Let Y be a set and (Y) be an algebra of subsets of Y. Let $H_n \in \mathrm{Bos}(X)$ for a topological space X, $n \in \mathcal{N}$. Suppose further that $F : Y \to X$ is a set-valued mapping with $F(y)$ a nonempty compact set for every $y \in Y$ and such that $F^{-1}(H) \in (Y)$ for every closed set H in X.*

Then there is an $S : Y \to X$ such that

(a) *$S(y) \subset F(y)$ is a nonempty compact subset of X for every $y \in Y$,*
(b) *$S^{-1}(H_n) \cap S^{-1}(X \setminus H_n) = \emptyset$, and*
(c) *$S^{-1}(H_n) \in (Y)$.*

This theorem has couple of corollaries. Let us recall that a Tychonoff space X is termed to be of *absolute class $\mathcal{F}$*, if X is of class $\mathcal{F}$ in any Tychonoff space it is embedded in.

Theorem 2. *For a space X, the following assertions are equivalent.*

(i) *The space X is of absolute class $\mathcal{F}$.*
(ii) *The space X is of class $\mathcal{F}$ in any compact space it is embedded in.*
(iii) *The space X is of class $\mathcal{F}$ in its Čech–Stone compactification.*

Another consequence of Theorem 1 is the following result on preservation of absolute classes under perfect mappings. We recall that a mapping $\varphi\colon X \to Y$ is *perfect* if it is continuous, closed and has compact fibers, i.e., $\varphi^{-1}(y)$ is compact for each $y \in Y$.

Theorem 3. *Let $\varphi\colon X \to Y$ be a perfect mapping of a space X of absolute class $\mathcal{F}$ onto a Tychonoff space Y. Then Y is of absolute class $\mathcal{F}$.*

We recall that X is *scattered-K-analytic* if $X \in \mathcal{S}(\mathrm{Hs}(\beta X))$ (here βX stand for the Čech–Stone compactification of X). A particular case of the previous theorem is a proof of the following conjecture of Hansell (see [10, Theorem 6.29]).

Theorem 4. *A perfect image of a scattered-K-analytic space is a scattered-K-analytic space.*

2.2. **Summary of Section 2.1: Extending Baire one functions on topological spaces.** The extension of mappings in a way started by the Tietze theorem stating that any continuous function on a closed subset of a normal space can be extended to a continuous function on the whole space. For Baire-1 mappings there is a classical result that any Baire-1 function on a G_δ subset of a metric space can be extended to a Baire-1 function defined on the whole space. The aim of this paper is an investigation of possibility of extending Baire-1 functions from subsets of topological spaces.

The first result is the following.

Theorem 5. *Let X be a space, $Y \subset X$ and*

(a) *Y is a cozero subset of X, or*

(b) *Y is its Lindelöf G_δ–subset.*

Then for any Baire-1 function f on Y there is a Baire-1 function g on X such that $f = g$ on Y,

$$\inf f(Y) = \inf g(X) \quad \textit{and} \quad \sup f(Y) = \sup g(X)\ .$$

To prove the main result we need the following key lemma on separating disjoint Lindelöf sets.

Lemma 3. *Let A and B be a couple of disjoint Lindelöf subsets of a space X.*

If there is no Coz_δ *set* G *satisfying* $A \subset G \subset X \setminus B$*, then there exists a nonempty closed set* $H \subset X$ *such that* $\overline{H \cap A} = \overline{H \cap B} = H$*.*

With the help of this key lemma one can proves the main result on extending Baire-1 functions. (We recall that a space X is *hereditarily Baire* if any closed subset of X is a Baire space.)

Theorem 6. *Let* Y *be a Lindelöf hereditarily Baire subset of a space* X *and* f *be a Baire-*1 *function on* Y*. Then there exists a Baire-*1 *function* g *on* X *such that* $f = g$ *on* Y*,*

$$\inf f(Y) = \inf g(X) \quad \textit{and} \quad \sup f(Y) = \sup g(X)\ .$$

Let $(F \vee G)_\delta$ stand for countable intersections of sets of the form $F \cup G$, where F is closed and G open. Since any $(F \vee G)_\delta$–subset of a hereditarily Baire space is also hereditarily Baire, we get the following corollary.

Theorem 7. *Let* Y *be a Lindelöf* $(F \vee G)_\delta$*–subset of a hereditarily Baire space* X*. Then any Baire-*1 *function* f *on* Y *can be extended to a Baire-*1 *function* g *on* X *so that* $f = g$ *on* Y*,*

$$\inf f(Y) = \inf g(X) \quad \textit{and} \quad \sup f(Y) = \sup g(X)\ .$$

Once we can extend Baire-1 functions, the natural question is whether there is a possibility of extending Baire-1 mappings with values in metrizable spaces. However, we cannot in general hope to find mappings which are pointwise limits of continuous functions, but at least we are able to extend mappings of the first Borel class, i.e., mappings that are $\Sigma_2(\mathrm{Bos}(X))$-measurable.

Theorem 8. *Let* Y *be a Lindelöf subset of a space* X*. Assume that*

(a) Y *is hereditarily Baire, or*
(b) *every* Coz_δ*–set in* X *is Lindelöf and* Y *is an* H*–set, or*
(c) Y *is* G_δ*–set in* X*.*

Then for any mapping $f : Y \to P$ *of the first Borel class to a Polish space* P *there exists a mapping* $g : X \to P$ *of the first Borel class such that* $f = g$ *on* Y *and* $g(X) \subset \overline{f(Y)}$*.*

The last main result is important for Section 3.1 and it reads as follows.

Theorem 9. *Let X be a compact set in a locally convex space such that $\operatorname{ext} X$ is Lindelöf. Let f be a Baire-1 function on $\operatorname{ext} X$. Then there exists a Baire-1 function g on X such that $f = g$ on X,*

$$\inf f(\operatorname{ext} X) = \inf g(X) \quad \textit{and} \quad \sup f(\operatorname{ext} X) = \sup g(X)\ .$$

2.3. **Summary of Section 2.1: Borel sets and functions in topological spaces.** The aim of this paper is to present an exposition of Borel classes in topological space and investigate their basic properties. First result was already mentioned in Section 2.1. It connects the measurability of mappings with the possibility of their pointwise approximation, i.e., the classical Lebesgue–Hausdorff–Banach theorem (see [19, Theorem 24.3]).

Theorem 10. *Let $\mathcal{F}$ be an algebra of sets in a set X and let Y be a separable metrizable space. Let Φ_1 stand for the family of all $\Sigma_2(\mathcal{F})$–measurable mappings from X to Y and, for $\alpha \in (1, \omega_1)$, let Φ_α be defined from Φ_1 as in Section 2.1.*

Then, for each $\alpha \in (0, \omega_1)$ and $f : X \to Y$, the following assertions are equivalent:

(i) $f \in \Phi_\alpha$,
(ii) *f is $\Sigma_{\alpha+1}(\mathcal{F})$–measurable.*

The basic properties of Borel sets in topological spaces are summarized in the following result.

Theorem 11. *Let X be a space. Then the following assertions hold:*

(a) $\Sigma_\alpha(\mathrm{Bas}(X)) \subset \Sigma_\alpha(\mathrm{Bos}(X)) \subset \Sigma_\alpha(\mathrm{Hs}(X))$, $\alpha \in (0, \omega_1)$,
(b) $\bigcup_{\alpha<\omega_1} \Sigma_\alpha(\mathrm{Bas}(X))$ *is the σ–algebra of all Baire sets in X and the family* $\bigcup_{\alpha<\omega_1} \Sigma_\alpha(\mathrm{Bos}(X))$ *is the σ–algebra of all Borel sets in X,*
(c) *if A is a subset of a normal space, then $A \in \Delta_2(\mathrm{Bas}(X))$ if and only if A is both F_σ and G_δ,*
(d) *if X is metrizable, then*
  (d1) $\Sigma_\alpha(\mathrm{Bas}(X)) = \Sigma_\alpha(\mathrm{Bos}(X))$, $\alpha \in (0, \omega_1)$,
  (d2) $\Sigma_\alpha(\mathrm{Hs}(X)) = \Sigma_\alpha(\mathrm{Bos}(X)) = \Sigma^0_\alpha(\mathcal{G}(X))$, $\alpha \in (1, \omega_1)$,
  (d3) *if X is completely metrizable, then* $\mathrm{Hs}(X) = \Delta_2(\mathrm{Bos}(X))$.

The results of Section 2.1 are then used in a partial answer to a question by Mauldin (see [26]). First we recall the notion of order.

If $\mathcal{F}$ is a family of sets in a set X, the *order of* $\mathcal{F}$ (denoted as $\operatorname{ord}(\mathcal{F})$) is the least $\alpha \in (0, \omega_1)$ such that $\Sigma_\alpha(\mathcal{F}) = \Sigma_{\alpha+1}(\mathcal{F})$ if such α exists, otherwise the order is ω_1 (see [26, p. 433]).

If X is a topological space, we call $\operatorname{ord}(\mathrm{Bas}(X))$, $\operatorname{ord}(\mathrm{Bos}(X))$ and $\operatorname{ord}(\mathrm{Hs}(X))$ the *Baire, Borel and resolvable order of* X, respectively.

If X is a compact space, it is well known that $\operatorname{ord}(\mathrm{Bas}(X))$ is either ω_1 or smaller or equal than 2, depending on the fact whether X is scattered or not (see [5], [28, Theorem 3.4], [30, Theorem 6.1.2] or Theorem 12 below). The question of the possible values of the Borel order of a compact space X is asked in [26, Question, p. 440] and [27, Problem, p. 295]. The following theorem solves one part of this problem.

Theorem 12. *For a space X the following assertions hold.*

(a) *If X is a K–analytic σ–scattered space, then*

$$\operatorname{ord}(\mathrm{Bas}(X)) \leq 2 \quad \textit{and} \quad \operatorname{ord}(\mathrm{Hs}(X)) \leq 2.$$

(b) *If X contains a compact perfect set, then*

$$\operatorname{ord}(\mathrm{Bas}(X)) = \operatorname{ord}(\mathrm{Bos}(X)) = \operatorname{ord}(\mathrm{Hs}(X)) = \omega_1.$$

However, the general question is still open.

Question. *Let X be a compact scattered space. Is it true that*

$$\operatorname{ord}(\mathrm{Bos}(X)) \leq 2?$$

The last part of the paper connects Borel measurable mappings with pointwise limits of sequences of continuous functions. First we define classes created by means of pointwise limits of Borel measurable mappings.

We consider the following classes of mappings between topological spaces X and Y.

(a) Let $\mathrm{Baf}_1(X, Y)$ be the family of all $\Sigma_2(\mathrm{Bas}(X))$–measurable mappings from X to Y and for $\alpha \in (1, \omega_1)$, let

$$\mathrm{Baf}_\alpha(X, Y) = (\mathrm{Baf}_1(X, Y))_\alpha.$$

We call the elements of $\bigcup_{\alpha<\omega_1} \mathrm{Baf}_\alpha(X, Y)$ the *Baire measurable mappings.*

(b) Let $\mathrm{Bof}_1(X, Y)$ be the family of all $\Sigma_2(\mathrm{Bos}(X))$–measurable mappings from X to Y and for $\alpha \in (1, \omega_1)$, as above we set

$$\mathrm{Bof}_\alpha(X, Y) = (\mathrm{Bof}_1(X, Y))_\alpha.$$

We call the elements of $\bigcup_{\alpha<\omega_1} \mathrm{Bof}_\alpha(X,Y)$ the *Borel measurable mappings.*

(c) Let $\mathrm{Hf}_1(X,Y)$ be the family of all $\Sigma_2(\mathrm{Hs}(X))$–measurable mappings from X to Y and for $\alpha \in (1,\omega_1)$, as above we set

$$\mathrm{Hf}_\alpha(X,Y) = (\mathrm{Hf}_1(X,Y))_\alpha.$$

We call the elements of $\bigcup_{\alpha<\omega_1} \mathrm{Hf}_\alpha(X,Y)$ the *resolvably measurable mappings.*

The following theorem justifies the term "measurability" in the definition above.

Theorem 13. *Let f be a mapping from a Tychonoff space X to a separable metrizable space Y and $\alpha \in (0,\omega_1)$. Then the following assertions hold:*

(a) *$f \in \mathrm{Baf}_\alpha(X,Y)$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Bas}(X))$–measurable.*
(b) *$f \in \mathrm{Bof}_\alpha(X,Y)$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Bos}(X))$–measurable.*
(c) *$f \in \mathrm{Hf}_\alpha(X,Y)$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$–measurable.*

Let now recall the classical Baire classes of mappings between topological spaces.

Let $\alpha \in (0,\omega_1)$. A mapping $f : X \to Y$ between topological spaces X and Y is said to be of *Baire class* α if $f \in (\mathcal{C}(X,Y))_\alpha$, where $\mathcal{C}(X,Y)$ denotes the set of all continuous mappings from X to Y. We write $\mathcal{C}_\alpha(X,Y)$ for the family of all mappings of Baire class α.

The following theorem is a variant of the classical characterization of mappings of Baire class α via their measurability (see e.g. [19, Theorem 24.3] or [8, Theorem 3]).

Theorem 14. *Let X be a compact space, Y be an arcwise connected locally arcwise connected metric space Y and $f : X \to Y$ be a function such that $f^{-1}(U)$ is a Baire subset of X for every open $U \subset Y$. Let $\alpha \in (0,\omega_1)$. Then the following assertions are equivalent:*

(i) *$f \in \mathrm{Baf}_\alpha(X,Y)$,*
(ii) *$f \in \mathrm{Bof}_\alpha(X,Y)$,*
(iii) *$f \in \mathrm{Hf}_\alpha(X,Y)$,*
(iv) *$f \in \mathcal{C}_\alpha(X,Y)$.*

The most difficult part of the proof of Theorem 14 is to show that $\mathcal{C}_1(X,Y)$ equals the space of $\Sigma_2(\mathrm{Bas}(X))$–measurable mappings. There is a long series of papers devoted to the question under what conditions a function $f : X \to Y$ is of Baire class 1 if and only if $f^{-1}(U)$ is F_σ for each $U \subset Y$ open. This question has an affirmative answer in any of the following situations:

- X is an interval in $\mathbb{R}$ and $Y = \mathbb{R}$ (see [2]),
- X is metric, $Y = \mathbb{R}$ (see [23]),
- X is metric, $Y = [0, 1]^n$, $n \in \mathbb{N}$, or $Y = [0, 1]^{\mathbb{N}}$ (see [21, §27, IX]),
- X is metric, Y is a separable convex subset of a Banach space (see [31, Lemma 3]),
- X is a complete metric space and Y is a Banach space (see [33, Theorem 4]),
- X is a normal topological space, $Y = \mathbb{R}$ (see [9] or [25, Exercise 3.A.1]),

If $f : X \to Y$ is *σ–discrete* (see [9, §3], [14, Section 2.2] or [34, p. 144] for the definition and basic properties), then f is of Baire class 1 if and only if $f^{-1}(U)$ is F_σ for each $U \subset Y$ open in any of the following situation:

- X is a perfectly normal paracompact space, Y is a Banach space (see [13, Corollary 7]),
- X is collectionwise normal and Y is a closed convex subset of a Banach space (see [9]),
- X is metric, Y is a complete connected and locally connected metric space (see [8, Theorem 2]),
- X is normal, Y is arcwise connected and locally arcwise connected and f is strongly σ-discrete (see [34, Theorem 3.7]).

3. Summary of Chapter 3

3.1. **Summary of Section 3.1.: A solution of the abstract Dirichlet problem for Baire one functions.** The first section of this chapter deals with the abstract Dirichlet problem for Baire-1 functions on compact convex sets. If X is a compact convex set in a locally convex space, the classical Krein-Milman theorem asserts that $X = \overline{\operatorname{conv}} \operatorname{ext} X$, i.e., X equals the closed convex hull of the set $\operatorname{ext} X$ of all extreme points of X. This theorem can be reformulated in the following way. For any point $x \in X$ there exists a measure $\mu \in \mathcal{M}^1(X)$ such that $x = \int_X \operatorname{id} d\mu$ (here $\mathcal{M}^1(X)$ stands for the Radon probability measures on X and

$\int_X \mathrm{id}\; d\mu$ is the Pettis integral from the identity mapping.) The equality $x = \int_X \mathrm{id}\; d\mu$ can be reformulated as follows: for any $h \in \mathfrak{A}^c(X)$ it holds $h(x) = \int_X h\, d\mu$. We call such a measure μ a *representing* measure for x.

The question of finding a representing measure that is more or less carried by the set $\operatorname{ext} X$ leads to the definition of the Choquet order. We say that $\mu, \nu \in \mathcal{M}^1(X)$ satisfy $\mu \preceq \nu$ if $\mu(k) \leq \nu(k)$ for any convex continuous function on X. Then for any $x \in X$ there exists a $\preceq$-maximal measure μ representing x that is carried by any Baire set containing $\operatorname{ext} X$ (this is the content of the Choquet–Bishop–de-Leeuw theorem, see [24, Section 3.8]). If the maximal representing measures are uniquely determined, X is said to be a *simplex*. In a finite dimensional space, this notion leads to the classical definition of a simplex. If X is a simplex, let δ_x denotes the maximal measure representing $x \in X$.

The abstract Dirichlet problem is a question whether it is possible to extend a given function on $\operatorname{ext} X$ to an affine function on X.

Bauer showed in [3] that X is a simplex with $\operatorname{ext} X$ closed if and only if any bounded continuous function on $\operatorname{ext} X$ can be extended to a function $h \in \mathfrak{A}^c(X)$. The content of Section 3.1. is a proof of a conjecture due to Jellett (see [16]) which was further elaborated by Kalenda (see [18]). The main result is the following.

Theorem 15. *Let X be a compact convex set. Then the following assertion are equivalent:*

(i) *X is a simplex and $\operatorname{ext} X$ is a Lindelöf H–set,*

(ii) *X is a simplex and for any closed G_δ set $F \subset X$ the function $x \mapsto \delta_x(F)$, $x \in X$, is Baire–one,*

(iii) *X is a simplex and the function $x \mapsto \delta_x(f)$, $x \in X$, is Baire-1 for every bounded Baire-1 function f on X,*

(iv) *for every bounded Baire-1 function f on X there exists an affine Baire-1 function h on X such that $f = h$ on $\operatorname{ext} X$,*

(v) *for every bounded Baire-1 function f on $\operatorname{ext} X$ there exists an affine Baire-1 function h on X such that $f = h$ on $\operatorname{ext} X$.*

Among the main ingredients of the proof belong Theorem 9 and results of Section 2.1.

3.2. **Summary of Section 3.2: Descriptive properties of elements of biduals of Banach spaces.** The second application of the results of Section 2.1 deals with the descriptive properties of affine functions on compact convex sets. We formulate our results in the language of Banach spaces and their duals. We recall that a function on a compact convex set X is *strongly affine* if $f(x) = \mu(f)$ for every $x \in X$ and every measure $\mu \in \mathcal{M}^1(X)$ representing x.

Now we can state our generalization of Saint Raymond's result contained in [32, Corollaire 8].

Theorem 16. *Let E be a Banach space and $f \in E^{**}$ be strongly affine. Then,*

- *for $\alpha \in [1, \omega_1)$, $f|_{\overline{\operatorname{ext} B_{E^*}}} \in \mathrm{Hf}_\alpha(\overline{\operatorname{ext} B_{E^*}})$ if and only if $f \in \mathrm{Hf}_\alpha(B_{E^*})$,*
- *for $\alpha \in [1, \omega_1)$, $f|_{\overline{\operatorname{ext} B_{E^*}}} \in \mathrm{Bof}_\alpha(\overline{\operatorname{ext} B_{E^*}})$ if and only if $f \in \mathrm{Bof}_\alpha(B_{E^*})$,*
- *for $\alpha \in [0, \omega_1)$, $f|_{\overline{\operatorname{ext} B_{E^*}}} \in \mathcal{C}_\alpha(\overline{\operatorname{ext} B_{E^*}})$ if and only if $f \in \mathcal{C}_\alpha(B_{E^*})$.*

Further we focus on the case when the set of extreme points is Lindelöf.

Theorem 17. *Let E be a Banach space such that $\operatorname{ext} B_{E^*}$ is a Lindelöf set. Let $f \in E^{**}$ be a strongly affine element satisfying $f|_{\operatorname{ext} B_{E^*}} \in \mathcal{C}_\alpha(\operatorname{ext} B_{E^*})$ for some $\alpha \in [0, \omega_1)$. Then*

$$f \in \begin{cases} \mathcal{C}_{\alpha+1}(B_{E^*}), & \alpha \in [0, \omega_0), \\ \mathcal{C}_\alpha(B_{E^*}), & \alpha \in [\omega_0, \omega_1). \end{cases}$$

By assuming a stronger assumption on $\operatorname{ext} B_{E^*}$ we may ensure the preservation of all classes, including the finite ones.

Theorem 18. *Let E be a Banach space such that $\operatorname{ext} B_{E^*}$ is a resolvable Lindelöf set. Let $f \in E^{**}$ be a strongly affine element satisfying $f|_{\operatorname{ext} B_{E^*}} \in \mathcal{C}_\alpha(\operatorname{ext} B_{E^*})$ for some $\alpha \in [1, \omega_1)$. Then $f \in \mathcal{C}_\alpha(B_{E^*})$.*

For a particular class of Banach spaces, namely the L_1-preduals, one can obtain some information on the affine class of a function from its descriptive class (we recall that a Banach space E is an *L_1-predual* if E^* is isometric to some space $L_1(\mu)$; see [17, p. 59], [22, Chapter 7] or [11, Section II.5]). Affine classes $\mathfrak{A}_\alpha(X)$, $\alpha < \omega_1$, of functions on a compact convex set X are created inductively from $\mathfrak{A}_0(X) = \mathfrak{A}^c(X)$ (see [4] or [24, Definition 5.37]). We also remark that a pointwise convergent sequence of affine functions on X is uniformly bounded which easily follows from the uniform boundedness principle (see e.g. [24, Lemma 5.36]), and thus any function in $\bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(X)$ is strongly affine. If $X = B_{E^*}$ is the dual unit ball of a Banach space E, the affine classes are termed *intrinsic Baire classes* of E in [1, p. 1047] whereas strongly affine Baire functions on X creates hierarchy of *Baire classes* of E. Theorem 19 relates these classes for real L_1-preduals.

We recall that, given a compact convex set X in a real locally convex space, the Banach space $\mathfrak{A}^c(X)$ is an L_1-predual if and only if X is a *simplex* (see [7, Theorem 3.2 and Proposition 3.23]).

Theorem 19. *Let E be a L_1-predual and $f \in E^{**}$ be a strongly affine function such that $f \in \mathcal{C}_\alpha(B_{E^*})$ for some $\alpha \in [2, \omega_1)$. Then*

$$f \in \begin{cases} \mathfrak{A}_{\alpha+1}(B_{E^*}), & \alpha \in [2, \omega_0), \\ \mathfrak{A}_\alpha(B_{E^*}), & \alpha \in [\omega_0, \omega_1). \end{cases}$$

If, moreover, $\operatorname{ext} B_{E^*}$ *is a Lindelöf resolvable set, then $f \in \mathfrak{A}_\alpha(B_{E^*})$.*

4. List of articles (sections)

The thesis is formed by the five articles contained in the following list. (The symbol IF denotes the value of the impact factor of the corresponding journal in year 2013. The list of citations of each article is typed in a small font.)

Sec. 2.1: Petr Holický and Jiří Spurný, *Perfect images of absolute Souslin and absolute Borel Tychonoff spaces,* Topology and its Applications **131** (2003), 181-194. (IF=0.562)

- [**Q1**] R. Pol, Evaluation maps on products of separable metrizable spaces are Borel, *Houston J. Math.*, 32(4) (2006), 1191–1196.
- [**Q2**] A. V. Ostrovskii, Mappings of Borel sets, (Russian) *Tr. Mat. Inst. Steklova* 252 (2006), *Geom. Topol., Diskret. Geom. i Teor. Mnozh., 237–260; translation in Proc. Steklov Inst. Math.* 2006, 252(1), 225–247.
- [**Q3**] O. Okunev, A relation between spaces implied by their t-equivalence. *Topology Appl.*, 158 (2011), no. 16, 2158-˘2164.
- [**Q4**] P. Holický, J. Pelant, Internal descriptions of absolute Borel classes. *Topology Appl.*, 141 (2004), no. 1-3, 87–104.
- [**Q5**] P. Holický, Preservation of completeness by some continuous maps. *Topology Appl.*, 157 (2010), no. 12, 1926–1930.
- [**Q6**] P. Holický, Descriptive classes of sets in nonseparable spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 104 (2010), no. 2, 257–282.

Sec. 2.2: Ondřej Kalenda and Jiří Spurný, *Extending Baire one functions on topological spaces,* Topology and its Applications **149(1-3)** (2005), 195-216. (IF=0.562)

- [**Q1**] D. H. Leung, W. K. Tang, Extension of functions with small oscillation. *Fund. Math.*, 192 (2006), no. 2, 183–193.
- [**Q2**] O. Karlova, Extension of continuous mappings and H1-retracts. *Bull. Aust. Math. Soc.*, 78 (2008), no. 3, 497–506.
- [**Q3**] M. Wojtowicz, W. Sieg, P-spaces and an unconditional closed graph theorem. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 104 (2010), no. 1, 13–18.
- [**Q4**] O. Karlova, Extension of continuous functions to Baire-one functions, *Real Anal. Exchange*, 36, no. 1 (2010), 149–160.
- [**Q5**] O. Karlova, On alpha-embedded sets and extension of mappings, *Comment. Math. Univ. Carolin.* 54, no. 3 (2013), 377–396.

Sec. 2.3: Jiří Spurný, *Borel sets and functions in topological spaces,* Acta Mathematica Hungarica, **129(1-2)** (2010), 47-69. (IF=0.348)

- [**Q1**] V.R. Timofey, V.K. Zakharov, A fine correlation between Baire and Borel functional hierarchies, *Acta Math. Hungar.* published online September 2013.

Sec. 3.1: Jiří Spurný and Ondřej Kalenda, *A solution of the abstract Dirichlet problem for Baire one functions,* Journal of Functional Analysis **232(2)** (2006), 259-294. (IF=1.252)

Sec. 3.2: Pavel Ludvík and Jiří Spurný, *Descriptive properties of elements of biduals of Banach spaces,* Studia Mathematica, **209(1)** (2012), 71-99. (IF=0.549)

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